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Trajectories of a DAE near a pseudo-equilibrium

R E Beardmore[1], R Laister[2] and A Peplow[3]

[1] Department of Mathematics, Imperial College, South Kensington, University of London, London SW7 2AZ, UK
[2] School of Mathematical Sciences, University of the West of England, Frenchay Campus, Bristol, BS16 1QY, UK
[3] MWL, S-100 44 Stockholm Royal Institute of Technology, Sweden

E-mail: r.beardmore@ic.ac.uk, robert.laister@uwe.ac.uk and andrew@fkt.kth.se

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Abstract

We consider a class of differential-algebraic equations (DAEs) defined by analytic nonlinearities and study its singular solutions. The main assumption used is that the linearization of the DAE represents a Kronecker index-2 matrix pencil and that the constraint manifold has a quadratic fold along its singularity.

From these assumptions we obtain a normal form for the DAE where the presence of the singularity and its effects on the dynamics of the problem are made explicit in the form of a quasi-linear differential equation. Subsequently, two distinct types of singular points are identified through which there pass exactly two analytic solutions: pseudo-nodes and pseudo-saddles. We also demonstrate that a singular point called a pseudo-node supports an uncountable infinity of solutions which are not analytic in general.

Moreover, akin to known results in the literature for DAEs with singular equilibria, a degenerate singularity is found through which there passes one analytic solution such that the singular point in question is contained within a quasi-invariant manifold of solutions. We call this type of singularity a pseudo-centre and it provides not only a manifold of solutions which intersects the singularity, but also a local flow on that manifold which solves the DAE.

Mathematics Subject Classification:

1. Introduction

Consider the differential-algebraic equation (DAE)

$$\dot{x} = f(x, y), \tag{1}$$

$$g(x, y) = 0, \tag{2}$$

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where $x \in \mathbb{R}^n$ and $n \geqslant 2$ throughout this paper, $y \in \mathbb{R}^m$, $f : \mathcal{U} \to \mathbb{R}^n$ and $g : \mathcal{U} \to \mathbb{R}^m$ are analytic in an open neighbourhood, $\mathcal{U}$, of $(0, 0)$ in $\mathbb{R}^{n+m}$.

Let us begin by defining some basic terminology associated with (1) and (2). The set $\boldsymbol{C} := g^{-1}(0) \cap \mathcal{U}$ is the constraint manifold for (1) and (2) and the singularity is defined by $\boldsymbol{S} := \{(x, y) \in \boldsymbol{C}\colon \det(d_y g(x, y)) = 0\}$. A solution of (1) and (2) on an interval $I \subseteq \mathbb{R}$ (so that I is a connected set containing at least two points) is considered to be a function $(x(\cdot), y(\cdot)) \in C^1(I, \mathbb{R}^n) \times C^0(I, \mathbb{R}^m)$ with this differential equation satisfied everywhere on I and the solution is said to be singular if there is a $t \in I$ such that $(x(t), y(t)) \in \boldsymbol{S}$. With regard to the term singular and its meaning in the context of (1) and (2), we understand a DAE to be singular at $(x, y) \in \boldsymbol{C}$ (and (x, y) is said to be a singular point) when the rank of $d_y g(x, y)$ is not constant in any neighbourhood of (x, y). Precisely, there is a sequence $(x_n, y_n) \subset \boldsymbol{C}$ such that $\operatorname{rank}(d_y g(x_n, y_n)) \neq \operatorname{rank}(d_y g(x, y))$ and yet $(x_n, y_n) \to (x, y)$.

A trajectory of (1) and (2) is a mapping $z : I \to \boldsymbol{C}$ which solves (1) and (2) and the orbit corresponding to this trajectory is $\{z(t)\colon t \in I\}$. A point $z_0 \in \boldsymbol{C}$ is said to support a solution of (1) and (2) if it is contained in some orbit, in this instance we may also say that a given trajectory passes through z_0.

The behaviour of solutions of the ordinary differential equation

$$\dot{z} = F(z), \tag{3}$$

in the neighbourhood of a given point z_0, where F has a given degree of regularity, is well-known. Any solution $z(t)$ has the same degree of regularity as F and the local geometric properties of the flow induced by (3) are given by the flow-box theorem if $F(z_0) \neq 0$, and by the description of the invariant (centre, stable and unstable) manifolds if $F(z_0) = 0$, provided that the linearization $\mathrm{d}F(z_0)$ has certain spectral properties.

Such a full description of the behaviour of a DAE in the vicinity of a singular point does not yet exist in the literature and a recent survey of the methods of analysis and areas of application of singular DAEs is to be found in Rabier and Rheinboldt (2002, chapter VII). Similarly, in contrast with ordinary differential equations, it is not the case that solutions of a DAE necessarily respect the regularity of the functions defining that DAE, and non-smooth solutions of DAEs are studied in Rabier and Rheinboldt (1995). We shall seek to address some of the problems surrounding the regularity of solutions of (1) and (2) in this paper.

This work may be considered an extension of Venkatasubramanian *et al* (1995) whose authors observed that singular points could support smooth solutions and that invariant manifolds of smooth solutions could intersect the singularity of a DAE. Other studies which complement the findings of this paper can be found in Takens (1976), Rabier and Rheinboldt (1994), von Sosen (1994), Thomas (1997), Marszalek and Campbell (1997, 1999), Ren and Spence (1999), Bruce and Tari (2000), Riaza *et al* (2000), Riaza and Zufiria (2001), Sotomayor and Zhitomirskii (2001), all of which consider some aspect of singular DAEs. Many of the papers are concerned with either the behaviour of solutions near impasse points, or with the construction of a normal form for (1) and (2). This is usually a quasi-linear differential equation whose trajectories can be mapped to trajectories of (1) and (2); this paper is in the latter category and the preprint (Reißig and Boche 2001) gives a brief survey of results for DAEs near singular points obtained using a normal form approach. Other techniques have of course been employed in the literature and in (Seikkalä and Heikkila 1997) the authors study a class of implicit differential equations using comparison principles which permits the treatment of certain discontinuities.

The work of Venkatasubramanian *et al* cited earlier has been extended in Beardmore and Laister (2002), which considered how the singular solutions of (1) and (2) behave in the vicinity of a singular equilibrium, defined as follows.

Definition 1. *A singular equilibrium of (1) and (2) is a point* $(x, y) \in \boldsymbol{S}$ *such that* $f(x, y) = 0$.

Trivially, (1) and (2) admits a constant solution through a singular equilibrium and it was shown in Beardmore and Laister (2002), subject to further assumptions, that one can find two analytic solutions whose initial datum is that singular point. An elementary example of this is

$$\dot{x} = y, \qquad x^2 + y^2 = 1, \tag{4}$$

which admits the two solutions

$$x(t) = 1, \quad y(t) = 0 \qquad \text{and} \qquad x(t) = \cos(t), \quad y(t) = -\sin(t),$$

both of which satisfy $x(0) = 1$ and $y(0) = 0$. If we perturb (4) and instead consider, for instance,

$$\dot{x} = -\epsilon + y, \qquad x^2 + y^2 = 1, \tag{5}$$

which has the effect of taking the equilibrium point away from the singularity, then the only singular solutions of (5) terminate at impasse points. A thorough description of the nature of impasse points is given in Rabier and Rheinboldt (1994), and the basic property of an impasse point is that two solutions terminate at this singular point in either forwards or backwards time and that the derivative of the solution blows up at this point.

Less immediately, obvious perhaps, is that (4) admits Lipschitz solutions which can be obtained by concatenating the smooth solutions together:

$$(X(t), Y(t)) := \begin{cases} (1, 0), & t \leqslant 0, \\ (\cos(t), -\sin(t)), & t > 0, \end{cases}$$

is one such example. For higher-order problems one does not need singular equilibria in order to obtain Lipschitz solutions. To see this, consider the DAE

$$-\ddot{u} = 6v, \tag{6}$$

$$u = v - v^3. \tag{7}$$

Define two functions $v_+(t) = (1/\sqrt{3}) + t$ and $v_-(t) = (1/\sqrt{3}) - t$, and then set $u_\pm := v_\pm - v_\pm^3$. The pairs (u_+, v_+) and (u_-, v_-) are both analytic solutions of (6) and (7) that have the initial condition $(u(0), \dot{u}(0), v(0)) = ((1 - \sqrt{3})/\sqrt{3}, 0, 1/\sqrt{3})$, which is not a singular equilibrium point of (6) and (7). However, by concatenating the two given solutions $(u_+, \dot{u}_+, v_+)$ and $(u_-, \dot{u}_-, v_-)$ at the point $t = 0$ one can again construct Lipschitz solutions of (6) and (7). DAEs of the form (6) and (7) are of interest as they arise as the steady-state problem for the class of reaction–diffusion equation

$$u_t = u_{xx} + \lambda f(u, v), \qquad v_t = g(u, v), \tag{8}$$

where λ is a bifurcation parameter. The bifurcation structure of the steady-state problem of (8) when $(u, v) = (0, 0)$ is a singular equilibrium (so that $f(0, 0) = g(0, 0) = g_v(0, 0) = 0$) has been studied in (Beardmore and Laister 2003).

Let us continue with the following definition which is central to the development of this paper.

Definition 2. *Suppose that* $(x, y) \in \boldsymbol{S}$ *satisfies*

(A1) $N(d_y g(x, y)) = \langle k \rangle$, $k^{\mathrm{T}} k = 1$ *and* $N(d_y g(x, y)^{\mathrm{T}}) = \langle u \rangle$, $u^{\mathrm{T}} u = 1$,
(A2) $d_x g(x, y) d_y f(x, y) k \notin R(d_y g(x, y))$,
(A3) $d_{yy}^2 g(x, y)[k, k] \notin R(d_y g(x, y))$ *and*
(A4) $d(f \times g)(x, y) \in GL(\mathbb{R}^{n+m})$.

Then (x, y) *is called a folded singular point for (1) and (2). If, in addition to (A1)–(A4), the condition*

(A5) $d_x g(x, y) f(x, y) \in R(d_y g(x, y))$,

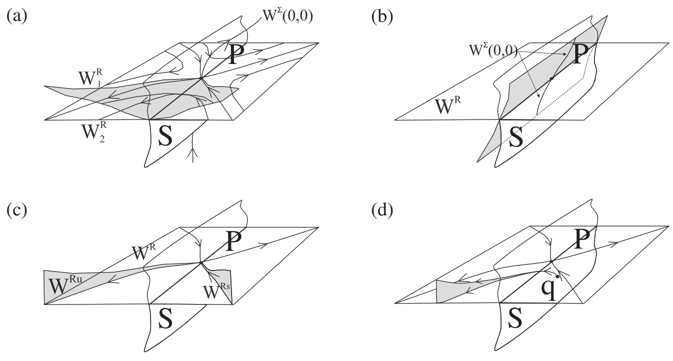

Figure 1. Invariant manifolds of (1) and (2) near a singular equilibrium (these are described in detail in Beardmore and Laister (2002)).

is satisfied, then (x, y) *is said to be a pseudo-equilibrium point for (1) and (2). If* $f(x, y) \neq 0$ *holds, in addition to (A1)–(A5), then* (x, y) *will be called a proper pseudo-equilibrium point and we shall denote by* $\boldsymbol{P} \subset \boldsymbol{S}$ *the set of pseudo-equilibria of (1) and (2).*

In Beardmore and Laister (2002) the nature of solutions of (1) and (2) at a singular equilibrium was analysed, subject to the restriction that it is also a pseudo-equilibrium point according to definition 2. The results of Beardmore and Laister (2002) are summarized in figure 1 where $\boldsymbol{C}$ is represented by $\mathbb{R}^3$ which is divided into two components by the singularity $\boldsymbol{S}$, which is itself divided into two components by $\boldsymbol{P}$. In figure 1(*a*) there is at least one manifold W^{R} (and possibly more) on which (1) and (2), subject to the stated assumptions, induces a local flow. Note that $W^{\mathrm{R}} \cap \boldsymbol{S} = \boldsymbol{P}$ so that elements of $\boldsymbol{P}$ are not impasse points, whereas points on $\boldsymbol{S} \backslash \boldsymbol{P}$ are impasse points. There is a further one-dimensional invariant manifold ($W^{\Sigma}(0, 0)$) which contains (0, 0) and which, subject to a further non-degeneracy requirement, has non-empty intersection with both components of $\boldsymbol{C} \backslash \boldsymbol{S}$. Associated with each $p \in \boldsymbol{P}$ there are also invariant manifolds $W^{\Sigma}(p)$ which intersect both components of $\boldsymbol{C} \backslash \boldsymbol{S}$. Figure 1(*c*) illustrates the fact that the existence of non-unique invariant manifolds (W_1^{R} and W_2^{R} from figure 1(*a*)) will ensure that the stable set associated with the equilibrium point (0, 0) does not have a manifold structure: certain subsets of the stable (W^{Rs}) and unstable (W^{Ru}) sets are illustrated. Similarly, figure 1(*d*) depicts the uniqueness of solutions of (1) and (2) breaking down at a point $q \in \boldsymbol{P}$ with a resulting continuum of solutions, however (Beardmore and Laister 2002) proffers no detailed explanation as to why this is the case.

Subsequently, throughout the remainder of the paper we shall be concerned with the situation where the point $(x, y) = (0, 0)$ is a proper pseudo-equilibrium point for (1) and (2). We shall show that one can associate a real, 2×2 matrix $\mathcal{Q}$ with this pseudo-equilibrium point whose eigenvalues (λ_1 and λ_2) determine the nature of the solutions of (1) and (2) through (0, 0) as follows:

(i) (Pseudo-focus) If λ_1 is complex (and therefore the conjugate of λ_2) with non-zero real and imaginary parts then there are no solutions.

(ii) (Pseudo-saddle) If the eigenvalues are real and $\lambda_1 \cdot \lambda_2 < 0$ then there are two Lipschitz and two analytic solutions.

(iii) (Pseudo-node) If the eigenvalues are real and $\lambda_1 \cdot \lambda_2 > 0$ then there are two analytic solutions and uncountably many solutions in $C^{\rho+1} \times C^{\rho}$, where ρ is the integer part of $\max\{\lambda_1/\lambda_2, \lambda_2/\lambda_1\}$, provided that the latter is not a natural number (the non-resonant case). If ρ is an integer then $C^2 \times C^1$ solutions can still be found, provided that $\lambda_1 \neq \lambda_2$.

(iv) (Pseudo-centre) If the eigenvalues are real and $\lambda_1\lambda_2 = 0$, $\lambda_1 + \lambda_2 \neq 0$ then, for any $k \in \mathbb{N}$, there is a codimension-1, C^k manifold of solutions of (1) and (2), $W^{\mathrm{R}} \subset \boldsymbol{C}$ which contains $(0, 0)$. There is also one analytic solution through $(0, 0)$.
(v) (Degenerate pseudo-centre) If $\sigma(\mathcal{Q}) = \{0\}$ then no information can be gleaned.

Assumptions (A1)–(A5) have the following interpretations. Assumption (A1) ensures that there is a non-empty constraint manifold and also ensures that the singularity $\boldsymbol{S}$ is non-empty. In fact, (A1)–(A4) are sufficient to ensure that $\boldsymbol{S}$ is a codimension-1, analytic submanifold of $\boldsymbol{C}$ which can be seen directly from the proof of Beardmore and Laister (2002, lemma 2.1). The simple null-space condition in (A1), along with (A2)–(A4), will allow us to use a Lyapunov–Schmidt reduction to find an n-dimensional normal form for (1) and (2) near the point $(x, y) = (0, 0)$. Assumption (A2) ensures that the linearization of (1) and (2) about $(x, y) = (0, 0)$ is an index-2 DAE of the form

$$M\dot{z} = (f(0, 0), 0) + Lz,$$

where, here and throughout $z := (x, y)$, M is the natural projection

$$M : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m, \qquad (x, y) \to (x, 0)$$

and $L = \mathrm{d}(f \times g)(0, 0)$ where d denotes the derivative. This means that the Kronecker index of the matrix pencil (M, L) is 2 (from Beardmore (2001, theorem 7)) and L is clearly non-singular by (A4).

Condition (A5) has a simple interpretation in terms of the existence of smooth solutions of (1) and (2) as follows. Suppose that $(x(t), y(t))$ is a differentiable solution with initial condition $(0, 0)$, differentiating (2) then gives $d_y g(0, 0)\dot{y}(0) = -d_x g(0, 0) f(0, 0)$ so that (A5) must be satisfied if $(0, 0)$ is to support a smooth solution passing through it. We have the following lemma which indicates precisely the nature of solutions through $(0, 0)$ if (A1)–(A4) are satisfied but (A5) is not:

Lemma 1.1. *Suppose that (x, y) is a folded singular point that is not a pseudo-equilibrium for (1) and (2), then it is an impasse point.*

Proof. This is an application of Rabier and Rheinboldt (1994, theorem 6.1). □

Assumptions (A1)–(A5), in particular (A2), are motivated directly by the singularity-induced bifurcation theorem of Venkatasubramanian *et al* (1995) and they are sufficient to ensure that $(x, y) = (0, 0)$ is an image-singularity (see Sotomayor and Zhitomirskii (2001, p 569)) for the quasi-linear differential equation obtained from (1) and (2) by differentiating (2) with respect to t. The classification scheme (i)–(v), devised according to the eigenvalues λ_1 and λ_2, is therefore similar to Sotomayor and Zhitomirskii (2001, theorem 3). However, the latter reference gives no indication of how the eigenvalues are related to the mappings f and g in (1) and (2), nor does it indicate what happens in the resonant cases of (iv). In addition, the pseudo-centre from part (iv) of the classification is not covered in either the references cited earlier.

2. Normal form for folded singular points

We continue with some notation and definitions. A set $K \subset \boldsymbol{C}$ is said to be quasi-invariant for (1) and (2) if for each $(x_0, y_0) \in K$ there exists a solution of (1) and (2), $(x(t), y(t))$ defined on $I \subset \mathbb{R}$, such that $(x(0), y(0)) = (x_0, y_0)$ and $(x(t), y(t)) \in K$ for all $t \in I$.We use the notion of quasi-invariance rather than invariance with regard to solutions of (1) and (2) because of the presence of the singularity and the subsequent non-uniqueness of solutions. For instance, equilibria of (4) are quasi-invariant but not invariant.

Let $I \subset \mathbb{R}$ be an open interval and N a natural number. We shall denote by $C^r(I, \mathbb{R}^N)$ the space of r-times continuously differentiable functions from I to $\mathbb{R}^N$. By $W^{1,\infty}(I, \mathbb{R}^N)$ we mean the space of weakly differentiable functions whose derivative lies in $L^\infty(I, \mathbb{R}^N)$. If a function u lies in $W^{1,\infty}(I, \mathbb{R})$ then u is absolutely continuous and $u(x) - u(y) = \int_y^x u'(s)\,\mathrm{d}s$, whence $|u(x) - u(y)| \leqslant |x - y| \|u\|_{W^{1,\infty}}$, where $\|u\|_{W^{1,\infty}} = \|u\|_{L^\infty} + \|u'\|_{L^\infty}$. It follows that u is Lipschitz on I.

We use $\mathcal{O}(n)$ to denote a function $h(\cdot)$ such that in some closed neighbourhood $\mathcal{W} \subset \mathbb{R}^N$ containing zero, there is a constant γ such that $\|h(w)\| \leqslant \gamma \|w\|^n$ for all w in $\mathcal{W}$. For $0 < x \in \mathbb{R}$, we shall use $\lfloor x \rfloor$ to denote the truncation of x to the nearest integer not greater than x. If F is any finite set then $\#F$ denotes its cardinality.

We use $\langle w_1, \ldots, w_m \rangle$ for the linear span of $w_1, \ldots, w_m$, $N(T)$ for the null-space of a linear map $T : \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{N}(T) = \bigcup_{k \geqslant 0} N(T^k)$ denotes the generalized null-space. Similarly, $R(T)$ denotes the range of T. If $S : \mathbb{R}^N \to \mathbb{R}^N$ is also a linear map, so that (S, T) is a matrix pencil on $\mathbb{R}^N$, then $\sigma(S, T) = \{\lambda \in \mathbb{C} : \det(\lambda S - T) = 0\}$ is its spectrum; we shall also write $\sigma(T)$ for the spectrum of T.

In order to reduce notational clutter, we shall not distinguish between 2×1 and 1×2 vectors, so that if for instance S is a 2×2 matrix, we may write $S \cdot (u, v)$ to denote the matrix–vector product which would normally be written as $S \cdot (u, v)^{\mathrm{T}}$.

Let us record the elements of the matrix $\mathrm{d}(f \times g)(0, 0)$ as follows:

$$L := \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} d_x f(0,0) & d_y f(0,0) \\ d_x g(0,0) & d_y g(0,0) \end{bmatrix} \tag{9}$$

and write L^{-1} using the block elements A_1, B_1, C_1 and D_1 in an analogous fashion:

$$L^{-1} := \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}. \tag{10}$$

This notation for the derivatives of f and g evaluated at the point $(x, y) = (0, 0)$ will be used throughout the paper. From Beardmore (2001) we have the following useful facts, recalling the definitions of u and k from (A1).

Lemma 2.1. *If $(x, y) = (0, 0)$ is a folded singular point for (1) and (2) and the mappings L and L^{-1} are defined according to (9) and (10), respectively, then*

$$\mathcal{N}(A_1) = \langle Bk \rangle, \qquad \mathcal{N}(A_1^{\mathrm{T}}) = \langle C^{\mathrm{T}} u \rangle, \qquad C_1 B k = k. \tag{11}$$

In addition, (M, L) is an index-2 matrix pencil and $\#\sigma(M, L) = n - 1$.

The proof of lemma 2.1 relies on all of the assumptions in (A1)–(A4) and the facts contained in (11) will be used throughout the derivation of a normal form for (1) and (2) at a folded singular point that is to follow. The non-zero constant

$$\omega := u^{\mathrm{T}} d_x g(0,0) d_y f(0,0) k = u^{\mathrm{T}} C B k \tag{12}$$

appears at several instants throughout the paper. Define the space

$$U := \langle C^{\mathrm{T}} u \rangle^{\perp} \subset \mathbb{R}^n \tag{13}$$

and suppose that the linear mappings

$$P_B : U \oplus \langle Bk \rangle \to \mathbb{R} \qquad \text{and} \qquad P_U : U \oplus \langle Bk \rangle \to U \tag{14}$$

are defined in such a way that $w \mapsto Bk \cdot P_B[w]$ is the projection of $\mathbb{R}^n = U \oplus \langle Bk \rangle$ onto $\langle Bk \rangle$ along U and $w \mapsto P_U[w] + Bk \cdot P_B[w]$ is the identity on $U \oplus \langle Bk \rangle$. Specifically, we have $P_B[w] = (1/\omega)u^{\mathrm{T}}Cw$ and $P_U[w] = (I - (1/\omega)u^{\mathrm{T}}C)w$.

There are several choices for the regularity of solutions that one could make with regard to (1) and (2) and in this paper we are interested in solutions for which y is at least continuous. If we impose regularity on a solution of (1) and (2) beyond continuity of y, we can rewrite (1) and (2) as a quasi-linear ODE in a neighbourhood of a folded singular point, as the following theorem shows.

Theorem 2.1. *Suppose that $(x, y) = (0, 0)$ is a folded singular point for (1) and (2). Then there is an analytic diffeomorphism $\chi : B' \to C'$, where $B' \subset U \times \mathbb{R}$ and $C' \subset \boldsymbol{C}$ are open neighbourhoods of zero, with the following properties.*

There is a linear mapping $L_0 \in GL(U)$ an element $c \in U$, a pair $(f_0, f_1) \in U \times \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that the map $t \mapsto (x(t), y(t))$ is a solution of (1) and (2) with $k^{\mathrm{T}}y(\cdot) \in W^{1,\infty}(I, \mathbb{R})$ if and only if $(x(t), y(t)) = \chi(\alpha(t), \beta(t))$, where (α, β) is a solution of the quasi-linear ODE

$$\text{(NF)}\ \begin{cases} \dot{\alpha} = f_0 + L_0\alpha + F(\alpha, \beta), \\ s(\alpha, \beta)\dot{\beta} = f_1 + c^{\mathrm{T}}\alpha + \gamma\beta + G(\alpha, \beta). \end{cases}$$

Here, a solution of (NF) *means that $(\alpha(\cdot), \beta(\cdot)) \in C^1(I, U) \times W^{1,\infty}(I, \mathbb{R})$ and* (NF) *is satisfied for a.e. $t \in I$. The function $F \times G : B' \to U \times \mathbb{R}$ is $\mathcal{O}(2)$ at zero, the function $s : B' \to \mathbb{R}$ is analytic,*

$$d_\alpha s(0, 0)[v] = -\frac{1}{\omega}u^{\mathrm{T}}d^2_{xy}g(0, 0)[v, k] \tag{15}$$

for all $v \in U$, moreover

$$s(0, 0) = 0 \qquad \text{and} \qquad d_\beta s(0, 0) = -\frac{1}{\omega}u^{\mathrm{T}}d^2_{yy}g(0, 0)[k, k] \neq 0. \tag{16}$$

If

$$\Sigma := \{(\alpha, \beta) \in B' : s(\alpha, \beta) = 0\}, \tag{17}$$

then $\chi(\Sigma) = \boldsymbol{S} \cap C'$. Also

$$f_0 = P_U[f(0, 0)] \qquad \text{and} \qquad f_1 = P_B[f(0, 0)]. \tag{18}$$

Proof. By assumptions (A1)–(A4) and lemma 2.1 we may write

$$x = \alpha + qBk \in U \oplus \langle Bk \rangle$$

and

$$y = r + \beta k \in \langle k \rangle^{\perp} \oplus \langle k \rangle.$$

We now proceed as in the proof of Beardmore and Laister (2002, theorem 2.2), using the implicit function theorem to solve the equation

$$g(\alpha + qBk, r + \beta k) = 0 \tag{19}$$

in some neighbourhood of $(x, y) = (0, 0) \in \mathbb{R}^{n+m}$. This is done by defining a mapping $\bar{g}$ by $\bar{g}(\alpha, q, r, \beta) := g(\alpha + qBk, r + \beta k)$, so that

$$d_{q,r}\bar{g}(\alpha, q, r, \beta)[\bar{q}, \bar{r}] = \bar{q}d_x g(\alpha, q, r, \beta)[Bk] + d_y g(\alpha, q, r, \beta)[\bar{r}], \tag{20}$$

for all $\bar{q} \in \mathbb{R}$ and $\bar{r} \in \langle k \rangle^{\perp}$. Setting $(\alpha, q, r, \beta) = (0, 0, 0, 0)$ in (20) we have $d_{q,r}\bar{g}(0, 0, 0, 0)[\bar{q}, \bar{r}] = \bar{q}CBk + D\bar{r}$. Assumption (A1) and the fact that $CBk \notin R(D)$

by (A2) ensures that $d_{q,r}\bar{g}(0,0,0,0)$ is a bijection. Hence (19) can be solved for $q = X(\alpha,\beta)$ and $r = Y(\alpha,\beta)$, where $B' \subset U \times \mathbb{R}$ is a neighbourhood of zero and

$$X : B' \to \mathbb{R} \qquad \text{and} \qquad Y : B' \to \langle k\rangle^{\perp} \subset \mathbb{R}^m$$

are analytic functions in this neighbourhood such that $X(0,0) = 0$, $Y(0,0) = 0$.

Now define

$$\chi(\alpha,\beta) := (\alpha + X(\alpha,\beta)Bk, Y(\alpha,\beta) + \beta k), \tag{21}$$

so that $\chi : B' \to \boldsymbol{C}$ and set $C' = \chi(B')$, finally write $g(x,y) = Cx + Dy + \gamma(x,y)$, where γ is analytic and $\mathcal{O}(2)$ at zero. Let us list the derivatives of the functions X and Y:

(i) $d_\alpha X(0,0) = 0$, $d_\alpha Y(0,0) = 0$,

(ii) $d_\beta X(0,0) = 0$, $d_\beta Y(0,0) = 0$, (so that $\chi : B' \to C'$ is a diffeomorphism satisfying $\chi(0,0) = (0,0)$),

(iii) $d^2_{\alpha\beta}X(0,0)[v,\lambda] = -(\lambda/\omega)u^{\mathrm{T}}d^2_{xy}g(0,0)[v,k]$,

(iv) $d^2_{\beta\beta}X(0,0)[\mu,\lambda] = -(\mu\lambda/\omega)u^{\mathrm{T}}d^2_{yy}g(0,0)[k,k] \neq 0$,

(v) $d^2_{pp}X(0,0)[v,v'] = -(1/\omega)u^{\mathrm{T}}d^2_{xx}g(0,0)[v,v']$,

for $v, v' \in U$, $\lambda, \mu \in \mathbb{R}$.

To obtain (i)–(v), note first of all that $g(\chi(\alpha,\beta)) \equiv 0$ and therefore by differentiating with respect to α we find

$$d_x g(\chi(\alpha,\beta))[I_U + d_\alpha X(\alpha,\beta)[\cdot]Bk] + d_y g(\chi(\alpha,\beta))[d_\alpha Y(\alpha,\beta)[\cdot]] \equiv 0,$$

whence

$$C(I_U + d_\alpha X(0,0)[\cdot]Bk) + D(d_\alpha Y(0,0)[\cdot]) = 0.$$

This implies $u^{\mathrm{T}}C + \omega d_\alpha X(0,0)[\cdot] = 0$ and subsequently

$$d_\alpha X(0,0)[v] = -\frac{1}{\omega}u^{\mathrm{T}}Cv \equiv 0, \qquad \forall v \in U = \langle C^{\mathrm{T}}u\rangle^{\perp}.$$

But then $Dd_\alpha Y(0,0)[v'] = 0$ so that $d_\alpha Y(0,0)[v'] \in \langle k\rangle$ and $d_\alpha Y(0,0)[v'] \in \langle k\rangle^{\perp}$ for all $v' \in U$ by the definition of Y, whence $d_\alpha Y(0,0)[v'] \equiv 0$. This proves (i). Similarly, differentiating $g(\chi(\alpha,\beta)) \equiv 0$ with respect to β gives $d_\beta X(0,0)[\cdot]CBk + Dd_\beta Y(0,0)[\cdot] = 0$ so that from (A2) we obtain $d_\beta X(0,0) = 0$ and $d_\beta Y(0,0)$ follows. This is property (ii).

Also,

$$\begin{aligned}
0 \equiv\ & d^2_{xx}g(\chi(\alpha,\beta))[I_U + d_\alpha X(\alpha,\beta)[\cdot]Bk, d_\beta X(\alpha,\beta)[\cdot]Bk] \\
& + d^2_{xy}g(\chi(\alpha,\beta))[I_U + d_\alpha X(\alpha,\beta)[\cdot]Bk, d_\beta Y(\alpha,\beta)[\cdot] + \cdot k] \\
& + d_x g(\chi(\alpha,\beta))[d^2_{\alpha\beta}X(\alpha,\beta)[\cdot,\cdot]Bk] \\
& + d^2_{yx}g(\chi(\alpha,\beta))[d_\alpha Y(\alpha,\beta)[\cdot], d_\beta X(\alpha,\beta)[\cdot]Bk] \\
& + d^2_{yy}g(\chi(\alpha,\beta))[d_\alpha Y(\alpha,\beta)[\cdot], k + d_\beta Y(\alpha,\beta)[\cdot]] \\
& + d_y g(\chi(\alpha,\beta))[d^2_{\alpha\beta}Y(\alpha,\beta)[\cdot,\cdot]]
\end{aligned}$$

and therefore

$$d^2_{xy}g(0,0)[v,\lambda k] + d^2_{\alpha\beta}X(0,0)[v,\lambda]CBk + Dd^2_{\alpha\beta}Y(0,0)[v,\lambda] = 0,$$

for all $(v,\lambda) \in U \times \mathbb{R}$; property (iii) now follows. Properties (iv) and (v) are proven in the same way.

We now suppose that (x,y) is a solution of (1) and (2) on an interval $I \subset \mathbb{R}$ with $y^{\mathrm{T}}k \in W^{1,\infty}(I,\mathbb{R})$. Since $x \in C^1(I,\mathbb{R}^n)$ and we can define functions α and q via $x = \alpha + qBk \in U \oplus \langle Bk\rangle$ where $\alpha = P_U[x]$ and $q = P_B[x]$ (with the maps P_U and

P_B defined in (14)), it follows that α and q are also C^1 functions. Also we can define r and β via $y = r + \beta k$, so it follows that $r \in W^{1,\infty}$ also.

Pre-multiplying (1) and (2) by L^{-1} we obtain the system

$$L^{-1} M \dot{z} = L^{-1}(f(0,0), 0)^{\mathrm{T}} + z + \mathcal{O}(2), \tag{22}$$

now write (22) in the form

$$A_1 \dot{x} = A_1 f(0,0) + x + \mathcal{O}(2), \tag{23}$$

$$C_1 \dot{x} = C_1 f(0,0) + y + \mathcal{O}(2). \tag{24}$$

Setting $f(0,0) = f_0 + f_1 Bk \in U \oplus \langle Bk \rangle$, from lemma 2.1 we have

$$A_1 f(0,0) = A_1 f_0, \qquad C_1 f(0,0) = C_1 f_0 + f_1 k.$$

Now let A_0 be the invertible map obtained by restricting A_1 to its invariant space U:

$$A_0 := A_1|_U : U \to R(A_1) = \langle C^{\mathrm{T}} u \rangle^{\perp} = U. \tag{25}$$

Using (23) we obtain

$$\begin{aligned} A_1 \dot{x} &= A_1(\dot{\alpha} + \dot{q} Bk) \\ &= A_1 \dot{\alpha} \qquad (26) \\ &= A_1 f(0,0) + x + \mathcal{O}(2) \\ &= A_1 f(0,0) + \alpha + q Bk + \mathcal{O}(2) \qquad (27) \end{aligned}$$

and projecting (26) and (27) onto $U = R(A_1)$ along $\langle Bk \rangle$, we obtain $A_1 \dot{\alpha} = A_1 f_0 + \alpha + \mathcal{O}(2)$, whence

$$\dot{\alpha} = f_0 + A_0^{-1} \alpha + \mathcal{O}(2).$$

Using (24) we find

$$\begin{aligned} k^{\mathrm{T}} C_1 \dot{x} &= k^{\mathrm{T}} C_1 (\dot{\alpha} + \dot{q} Bk) \\ &= k^{\mathrm{T}} C_1 (f_0 + A_0^{-1} \alpha + \mathcal{O}(2) + \dot{q} Bk) \\ &= k^{\mathrm{T}} C_1 f_0 + k^{\mathrm{T}} C_1 A_0^{-1} \alpha + \dot{q} + \mathcal{O}(2), \qquad (28) \\ k^{\mathrm{T}} C_1 \dot{x} &= k^{\mathrm{T}} (C_1 f(0,0) + y + \mathcal{O}(2)) \\ &= k^{\mathrm{T}} (C_1 f_0 + f_1 k + r + \beta k + \mathcal{O}(2)) \\ &= k^{\mathrm{T}} C_1 f_0 + f_1 + \beta + \mathcal{O}(2) \qquad (29) \end{aligned}$$

and equating (28) and (29) we obtain $k^{\mathrm{T}} C_1 A_0^{-1} \alpha + \dot{q} = f_1 + \beta + \mathcal{O}(2)$.

Summarizing the previous calculations, we have obtained a system of equations satisfied by α and β:

$$\begin{aligned} \dot{\alpha} &= f_0 + A_0^{-1} \alpha + \mathcal{O}(2), \\ k^{\mathrm{T}} C_1 A_0^{-1} \alpha + \dot{q} &= f_1 + \beta + \mathcal{O}(2), \end{aligned}$$

and there remains to obtain an expression for $\dot{q}$, that is $\mathrm{d}X(\alpha, \beta)/\mathrm{d}t$. Since $\beta = k^{\mathrm{T}} y \in W^{1,\infty}$ holds by assumption, it follows that $X(\alpha, \beta) \in W^{1,\infty}$ and

$$\begin{aligned} \dot{q} &= d_\alpha X(\alpha, \beta) \dot{\alpha} + d_\beta X(\alpha, \beta) \dot{\beta} \\ &= d_\alpha X(\alpha, \beta)[f_0 + A_0^{-1} \alpha + \mathcal{O}(2)] + d_\beta X(\alpha, \beta) \dot{\beta}, \end{aligned}$$

but since

$$d_\alpha X(0,0) = 0 \qquad \text{and} \qquad d^2_{pp} X(0,0)[v,v] = -\frac{1}{\omega} u^{\mathrm{T}} d^2_{xx} g(0,0)[v,v], \quad \forall v \in U,$$

we find

$$d_\alpha X(\alpha,\beta)[f_0 + A_0^{-1}\alpha + \mathcal{O}(2)] = d_\alpha X(0,0)[f_0 + A_0^{-1}\alpha] + d^2_{pp}X(0,0)[\alpha, f_0 + A_0^{-1}\alpha]$$
$$+ d^2_{\alpha\beta}X(0,0)[f_0 + A_0^{-1}\alpha, \beta] + \mathcal{O}(2)$$
$$= d^2_{pp}X(0,0)[\alpha, f_0] + \beta d^2_{\alpha\beta}X(0,0)[f_0, 1] + \mathcal{O}(2)$$
$$= \frac{-1}{\omega}(u^{\mathrm{T}} d^2_{xx} g(0,0)[\alpha, f_0] + \beta u^{\mathrm{T}} d^2_{xy} g(0,0)[f_0, k]) + \mathcal{O}(2).$$

If we now define $s(\alpha,\beta) := d_\beta X(\alpha,\beta)$, we obtain a system of differential equations satisfied by α and β:

$$\dot\alpha = f_0 + A_0^{-1}\alpha + \mathcal{O}(2),$$
$$s(\alpha,\beta)\dot\beta = f_1 + c^{\mathrm{T}}\alpha + \gamma\beta + \mathcal{O}(2),$$

where

$$c^{\mathrm{T}}\alpha \equiv -k^{\mathrm{T}} C_1 A_0^{-1}\alpha + \frac{1}{\omega} u^{\mathrm{T}} d^2_{xx} g(0,0)[f_0, \alpha], \qquad \forall \alpha \in U \tag{30}$$

and

$$\gamma = 1 + \frac{1}{\omega} u^{\mathrm{T}} d^2_{xy} g(0,0)[f_0, k]. \tag{31}$$

It follows from property (iv) earlier that $s(0,0) = 0$ and $s_\beta(0,0) \neq 0$.

In order to prove the claim that $\chi(B' \cap \Sigma) = \boldsymbol{S} \cap C'$ we differentiate the expression $g(\chi(\alpha,\beta)) \equiv 0$, giving

$$d_x g(\alpha + X(\alpha,\beta)Bk, Y(\alpha,\beta) + \beta k)[d_\beta X(\alpha,\beta)Bk]$$
$$+ d_y g(\alpha + X(\alpha,\beta)Bk, Y(\alpha,\beta) + \beta k)[k + d_\beta Y(\alpha,\beta)] \equiv 0.$$

Using assumptions (A1) and (A2), this implies that

$$s(\alpha,\beta) = 0 \Longleftrightarrow d_y g(\alpha + X(\alpha,\beta)Bk, Y(\alpha,\beta) + \beta k)[k + d_\beta Y(\alpha,\beta)] = 0$$
$$\Longleftrightarrow \chi(\alpha,\beta) \in \boldsymbol{S},$$

for $(\alpha,\beta) \in B'$, and the claim therefore follows.

Conversely, if a solution of (NF) is given with the regularity conditions in the statement of the theorem, then it is clear that this is mapped to a solution of (1) and (2) by the diffeomorphism χ. □

Definition 3. *A pair* $(\alpha,\beta) \in \Sigma$ *is said to be a pseudo-equilibrium for* (NF) *if* $f_1 + c^{\mathrm{T}}\alpha + \gamma\beta + G(\alpha,\beta) = 0$. *Throughout, we shall denote the set of pseudo-equilibria of* (NF) *by* P *and a pseudo-equilibrium is said to be proper if* $f_0 + L_0\alpha + F(\alpha,\beta) \neq 0$.

As a consequence, the zero element $(0,0) \in U \times \mathbb{R}$ is a (proper) pseudo-equilibrium point for (NF) if and only if $(0,0) \in \boldsymbol{C}$ is a (proper) pseudo-equilibrium point for (1) and (2). Subsequently, we state all the remaining results in terms of (NF) rather than (1) and (2). Now the set P may be empty if $f_1 \neq 0$, but if $f_1 = 0$ then (A5) is satisfied and P is certainly non-empty as it contains the zero element of $U \times \mathbb{R}$ and the following lemma, which follows from Rabier and Rheinboldt (1994, theorem 2.1), says that impasse points are the norm rather than the exception for elements of Σ.

Lemma 2.2. *Any element of* $\Sigma \backslash P$ *is an impasse point for* (NF).

In the remainder we shall call the first equation in (NF) the ordinary part and the second equation will be called the quasi-linear part. The set Σ defined in (17) will be described as the singularity of (NF).

Remark 1. If f_1 $(=(1/\omega)u^{\mathrm{T}}Cf(0,0)) \neq 0$ in equation (18) of theorem 2.1, then (A5) is violated and lemma 2.2 tells us that there is no Lipschitz (or smoother) solution of (NF) satisfying the initial condition $\alpha(0) = 0, \beta(0) = 0$. Note that if $f_0 = 0$ and $f_1 = 0$, theorem 2.1 reduces to Beardmore and Laister (2002, theorem 2.2).

Clearly, if a solution of (NF) satisfies $s(\alpha(T), \beta(T)) \neq 0$, then α and β are analytic in some neighbourhood of T. On the other hand, if the solution encounters the singularity Σ, this may cause a jump in the derivative of β and the fact that $\beta \in W^{1,\infty}$ accounts for this.

Before proceeding, we must define some notation that is used in the remainder of the paper. In the case that $(x, y) = (0, 0)$ is a proper pseudo-equilibrium of (1) and (2), so that (A5) applies and $f_0 \neq 0$, $f_1 = 0$ as a result, we define the two-dimensional vector space $V \subset U \times \mathbb{R}$ by

$$V := \langle (f_0, 0), (0, 1) \rangle \subset \langle C^{\mathrm{T}}u \rangle^{\perp} \times \mathbb{R} \tag{32}$$

and let $\Phi(\alpha, \beta)$ denote the analytic mapping $\Phi : B' \ (\subset U \times \mathbb{R}) \to \mathbb{R}^2$

$$\Phi(\alpha, \beta) = (s(\alpha, \beta), c^{\mathrm{T}}\alpha + \gamma\beta + G(\alpha, \beta)) \tag{33}$$

such that $\Phi(0, 0) = (0, 0)$. Let us also form the decomposition

$$U = \langle f_0 \rangle^{\perp} \oplus \langle f_0 \rangle, \tag{34}$$

recalling the fact that $f_0 = P_U[f(0,0)]$ which coincides with $f(0,0)$ when $f_1 = 0$, and also let us record the fact that throughout the remainder of this paper we shall make use of the orthogonal decomposition (34) and write

$$\alpha = \lambda f_0 + a \in \langle f_0 \rangle \oplus \langle f_0 \rangle^{\perp} = U, \tag{35}$$

where $\lambda \in \mathbb{R}$ and $a \in U$ with $a^{\mathrm{T}} f_0 = 0$.

We have the following lemma which gives a non-degeneracy condition to ensure that P has a manifold structure.

Lemma 2.3. *Suppose that* $(0, 0)$ *is a proper pseudo-equilibrium for* (NF) *and*

$$u^{\mathrm{T}}d^2_{xy}g(0,0)[f_0, k](\omega + u^{\mathrm{T}}d^2_{xy}g(0,0)[f_0, k]) \neq u^{\mathrm{T}}d^2_{yy}g(0,0)[k,k](-\omega k^{\mathrm{T}}C_1A_0^{-1}f_0 + u^{\mathrm{T}}d^2_{xx}g(0,0)[f_0, f_0]),$$

then P *is a codimension-1, analytic submanifold of* Σ *which contains* $(\alpha, \beta) = (0, 0)$.

Proof. Since $(0, 0)$ is a pseudo-equilibrium for (NF), then $f_1 = 0$ in (NF). Now the mapping Φ defined in (33) satisfies $\Phi(0, 0) = (0, 0)$ and the derivative of Φ at $(0, 0)$, $\mathrm{d}\Phi(0, 0) : U \times \mathbb{R} \to \mathbb{R}^2$ is then given by

$$\mathrm{d}\Phi(0,0) = \begin{bmatrix} d_\alpha s(0,0) & d_\beta s(0,0) \\ c^{\mathrm{T}} & \gamma \end{bmatrix}$$

and using (30) and (31) this is the matrix

$$\frac{1}{\omega}\begin{bmatrix} -u^{\mathrm{T}}d^2_{xy}g(0,0)[\cdot, k] & -u^{\mathrm{T}}d^2_{yy}g(0,0)[k,k] \\ -\omega k^{\mathrm{T}}C_1A_0^{-1} + u^{\mathrm{T}}d^2_{xx}g(0,0)[f_0, \cdot] & \omega + u^{\mathrm{T}}d^2_{xy}g(0,0)[f_0, k] \end{bmatrix}.$$

If we define the mapping $\Phi_0(\lambda, \beta, a) = \Phi(\lambda f_0 + a, \beta)$, where $a \in \langle f_0 \rangle^{\perp} \subset U$, then $\Phi_0(0, 0, 0) = 0$ and $d_{\lambda,\beta}\Phi_0(0, 0, 0) = (1/\omega)\mathcal{Q}$ where $\mathcal{Q}$ is the 2×2 matrix

$$\begin{aligned} \mathcal{Q} &:= \omega \begin{bmatrix} d_\alpha s(0,0)[f_0] & d_\beta s(0,0) \\ c^{\mathrm{T}} f_0 & \gamma \end{bmatrix} \\ &= \begin{bmatrix} -u^{\mathrm{T}}d^2_{xy}g(0,0)[f_0, k] & -u^{\mathrm{T}}d^2_{yy}g(0,0)[k,k] \\ -\omega k^{\mathrm{T}}C_1A_0^{-1}f_0 + u^{\mathrm{T}}d^2_{xx}g(0,0)[f_0, f_0] & \omega + u^{\mathrm{T}}d^2_{xy}g(0,0)[f_0, k] \end{bmatrix}, \end{aligned} \tag{36}$$

which has a non-zero determinant by assumption. The result now follows from the implicit function theorem which allows us to represent P locally as a graph with $\lambda = \lambda(a)$ and $\beta = \beta(a)$. □

The matrix $\mathcal{Q}$ in (36) is crucial for what is to follow, so let us record its eigenvalues which we assume for now to be real and define $\varsigma \in \mathbb{R}$ to be that number which satisfies

$$\{\varsigma, \omega - \varsigma\} = \sigma(\mathcal{Q}), \tag{37}$$

noting that $\mathrm{tr}(\mathcal{Q}) = \omega$. For brevity, we shall also write

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \tag{38}$$

throughout. Let us label the eigenvectors of $\mathcal{Q}$ as follows:

$$\mathcal{Q}\boldsymbol{u} = \varsigma \boldsymbol{u}, \qquad \mathcal{Q}\boldsymbol{v} = (\omega - \varsigma)\boldsymbol{v}, \tag{39}$$

for non-zero vectors $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$. Now define the matrix $\mathcal{P} = [\boldsymbol{u}|\boldsymbol{v}]$ so that $\mathcal{Q}\mathcal{P} = \mathcal{P}\mathcal{D}$, where $\mathcal{D}$ is the diagonal matrix of eigenvalues $\mathrm{diag}\{\varsigma, \omega - \varsigma\}$.

Lemma 2.4. *Suppose that* $(0, 0)$ *is a proper pseudo-equilibrium point for (NF), then* $u_1 v_1 \neq 0$.

Proof. Suppose that $(0, 1)^{\mathrm{T}}$ is an eigenvector of $\mathcal{Q}$, it immediately follows from (36) that $u^{\mathrm{T}} d^2_{yy} g(0, 0)[k, k] = 0$ but this contradicts (A3). □

3. Desingularizing (NF)

The preparatory stages with regard to reducing (NF) to a local normal form are almost complete. The goal in this section is to show that in a neighbourhood of the pseudo-equilibrium point $(\alpha, \beta) = (0, 0)$ associated with (NF), the study of the local dynamics can be reduced to a planar system of differential equations on the two-dimensional space V defined in (32), the orbits of (NF) are then obtained from suitable images of this flow.

So, let the variable τ represent a new timescale defined by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = s(\alpha(t), \beta(t)), \qquad t(\tau_0) = t_0. \tag{40}$$

If we abuse notation and write $\alpha(\tau) = \alpha(t(\tau))$ and $\beta(\tau) = \beta(t(\tau))$, we then obtain the following differential equation:

$$\alpha' = s(\alpha, \beta)(f_0 + L_0\alpha + F(\alpha, \beta)), \tag{41}$$

$$\beta' = c^{\mathrm{T}}\alpha + \gamma\beta + G(\alpha, \beta), \tag{42}$$

where a prime (′) denotes $\mathrm{d}/\mathrm{d}\tau$. Since $f_0 + L_0\alpha + F(\alpha, \beta) \neq 0$ holds at a proper pseudo-equilibrium point of (NF), proper pseudo-equilibria of (NF) are equilibria of (41) and (42) which are not themselves equilibria of (NF).

The strategy used to locate solutions of (NF) through a pseudo-equilibrium will be to obtain a one-dimensional invariant manifold of (41) and (42) which contains the corresponding equilibrium point of this differential equation and which is transverse to the singularity. This manifold will then be quasi-invariant for (NF) and the orientation of the flow of a solution of (NF) will be different from that of (41) and (42) where $s(\alpha, \beta) < 0$. While the arrival time of a solution of (41) and (42) at this equilibrium along such a manifold will not be finite, the arrival time for (NF) at the pseudo-equilibrium may well be. If so, one can then concatenate the two resultant solutions of (NF) together to enlarge the interval of existence of either solution so as to create an orbit of (NF) which passes through the singularity. Due to the infinite time of flight of solutions of (41) and (42) along an invariant manifold containing an equilibrium, we will need to take $\tau_0 = +\infty$ and $\tau_0 = -\infty$ in (40), depending on the orientation of the flow of (41) and (42).

Further, the choice of timescale for (40) is only defined up to a constant. One could use $\mu s(\alpha(t), \beta(t))$ in (40) to desingularize (NF) where $\mu \in \mathbb{R}$ is any non-zero constant, thus multiplying both of equations (41) and (42) by a factor μ. However, the results in the remainder of this paper do not depend upon the sign of μ as different signs just correspond to different orientations of the orbits of (41) and (42).

3.1. Pseudo-nodes

The first theorem that we present regarding singular solutions of (NF), and therefore (1) and (2), is the following which is shown pictorially in figure 2.

Theorem 3.1. *Suppose* $(0, 0)$ *is a proper pseudo-equilibrium point for* (NF) *and* $\mathcal{Q}$ *has real and distinct eigenvalues* ς *and* $\omega - \varsigma$ *such that* $\mathrm{sgn}(\varsigma) = \mathrm{sgn}(\omega - \varsigma)$. *If, in addition,*

(R) $\rho := \max\{\varsigma/(\omega - \varsigma), (\omega - \varsigma)/\varsigma\} \notin \mathbb{N}$,

then there are uncountably many solutions (α, β) *of* (NF) *in* $C^{\lfloor\rho\rfloor+1} \times C^{\lfloor\rho\rfloor}$ *with initial condition* $(\alpha(0), \beta(0)) = (0, 0)$. *There is also an analytic solution with the same initial datum. Consequently, there are uncountably many* $C^{\lfloor\rho\rfloor+1} \times C^{\lfloor\rho\rfloor}$ *solutions* (x, y) *of (1) and (2) and an analytic solution, all with initial condition* $(x(0), y(0)) = (0, 0)$.

In addition, if

$$\omega k^{\mathrm{T}} C_1 A_0^{-1} f_0 \neq u^{\mathrm{T}} d_{xx}^2 g(0, 0)[f_0, f_0] \tag{43}$$

then there is a second analytic solution of (NF) *through* $(0, 0)$. *Subsequently, there is a second analytic solution* (x, y) *of (1) and (2) with initial condition* $(x(0), y(0)) = (0, 0)$.

Proof. First of all note that

$$\begin{aligned} s(\alpha, \beta)(f_0 + L_0\alpha + F(\alpha, \beta)) &= s(\alpha, \beta) f_0 + \mathcal{O}(2) \\ &= (d_\alpha s(0, 0)[\alpha] + d_\beta s(0, 0)[\beta]) f_0 + \mathcal{O}(2) \\ &= -\frac{1}{\omega} f_0(u^{\mathrm{T}} d_{xy}^2 g(0, 0)[\alpha, k] + \beta u^{\mathrm{T}} d_{yy}^2 g(0, 0)[k, k]) + \mathcal{O}(2). \end{aligned} \tag{44}$$

Using the decomposition (35), let us write $\alpha = \lambda f_0 + a \in U$ with $a^{\mathrm{T}} f_0 = 0$ and $u^{\mathrm{T}} C f_0 = u^{\mathrm{T}} C a = 0$. Using (41) and (42) and (44) we obtain

$$\begin{aligned} \alpha' &= \lambda' f_0 + a' \\ &= s(\lambda f_0 + a, \beta)(f_0 + L_0[\lambda f_0 + a] + F(\lambda f_0 + a, \beta)) \\ &= -\frac{1}{\omega} f_0(u^{\mathrm{T}} d_{xy}^2 g(0, 0)[\lambda f_0 + a, k] + \beta u^{\mathrm{T}} d_{yy}^2 g(0, 0)[k, k]) + \mathcal{O}(2) \end{aligned} \tag{45}$$

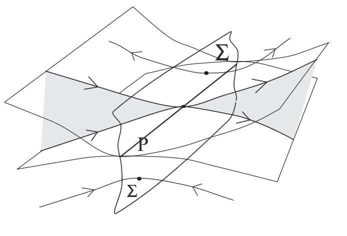

Figure 2. The flow of a DAE near a pseudo-node. A two-dimensional invariant manifold (a subset of which is indicated by the grey region) contains two analytic solutions and uncountably many solutions of finite smoothness which pass through the singularity. Singular points not on P are impasse points.

and taking the inner product of this with f_0 we find

$$\lambda' = -\frac{1}{\omega}(u^{\mathrm{T}} d^2_{xy} g(0,0)[\lambda f_0 + a, k] + \beta u^{\mathrm{T}} d^2_{yy} g(0,0)[k,k]) + \mathcal{O}(2).$$

Moreover, from (45) we also have $a' = \mathcal{O}(2)$, and from (42) there results $\beta' = \lambda c^{\mathrm{T}} f_0 + c^{\mathrm{T}} a + \gamma\beta + \mathcal{O}(2)$. If we define $\nu = (\lambda, \beta)$ then

$$\begin{bmatrix} \nu' \\ a' \end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} \mathcal{Q} & \mathcal{R} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \nu \\ a \end{bmatrix} + \mathcal{O}(2), \tag{46}$$

where the linear mapping $\mathcal{R} : \langle f_0 \rangle^{\perp} \to \mathbb{R}^2$ is given by

$$\mathcal{R}[a] = (-u^{\mathrm{T}} d^2_{xy} g(0,0)[a,k], -\omega k^{\mathrm{T}} C_1 A_0^{-1} a + u^{\mathrm{T}} d^2_{xx} g(0,0)[f_0, a]). \tag{47}$$

Finally, let us define

$$b := \mathcal{Q}\nu + \mathcal{R}a, \tag{48}$$

so that

$$\begin{bmatrix} b' \\ a' \end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} \mathcal{Q} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b \\ a \end{bmatrix} + \mathcal{O}(2). \tag{49}$$

Since $\mathrm{sgn}(\varsigma) = \mathrm{sgn}(\omega - \varsigma)$, if we assume for definiteness and without the loss of any generality that $\varsigma < 0$, reversing the direction of time if need be, we may assume without loss of generality that $\mathcal{Q}$ is a stable matrix, with two negative eigenvalues. Therefore the differential equation (49) has two invariant manifolds associated with its zero equilibrium, one a two-dimensional analytic stable manifold, $W^{\mathrm{s}}(0,0)$, described by a graph of the form

$$a = h(b), \qquad \text{with } h(0) = 0, \quad \mathrm{d}h(0,0) = 0 \tag{50}$$

and a centre manifold $W^{\mathrm{c}}(0,0)$, also described by a graph.

Let us now consider the stable manifold $W^{\mathrm{s}}(0,0)$ of (49). We can easily describe the invariant spaces of the linear differential equation

$$\omega b' = \mathcal{Q}b,$$

recalling $\mathcal{Q}\boldsymbol{u} = \varsigma\boldsymbol{u}$, $\mathcal{Q}\boldsymbol{v} = (\omega - \varsigma)\boldsymbol{v}$ and $\mathcal{P} = [\boldsymbol{u}|\boldsymbol{v}]$. On setting $\mathcal{P}\boldsymbol{w} = b$ with $\boldsymbol{w} = (\zeta, \eta)$ and b defined in (48) we obtain

$$\omega\boldsymbol{w}' = \mathcal{D}\boldsymbol{w}, \tag{51}$$

which has invariant manifolds described by the equation

$$(\omega - \varsigma)\frac{\mathrm{d}\zeta}{\zeta} = \varsigma\frac{\mathrm{d}\eta}{\eta}.$$

Let us now assume for definiteness and by (***R***) that

$$\rho = \frac{\varsigma}{\omega - \varsigma} > 1,$$

so that there are even and odd invariant manifolds

$$\zeta = p|\eta|^{\rho} \qquad \text{and} \qquad \zeta = p\eta|\eta|^{\rho-1}, \tag{52}$$

for some arbitrary constant p.

Denote the one-parameter family of even, $C^{\lfloor\rho\rfloor}$ manifolds arising in the (ζ, η) plane from (52) by $\mathcal{M}_p$, parameterized by $p \in \mathbb{R}$. Notice that $\mathcal{M}_0 = \{(\zeta, \eta) : \zeta = 0\}$ is analytic and there is a further analytic invariant manifold for (51) given by the set $\{(\zeta, \eta) : \eta = 0\}$, we shall call this set $\mathcal{M}$.

By the non-resonance assumption (***R***), there is an analytic linearizing diffeomorphism φ which carries the flow of the linear differential equation (51) to that of the nonlinear differential equation

$$\omega b' = \mathcal{Q}b + H(b), \tag{53}$$

where equation (53) denotes the restriction of (49) to the stable manifold $W^s(0, 0)$, thus defining the mapping H. The existence of an analytic diffeomorphism φ can be found in the works of Poincaré (1916–1954) and also in Sternberg (1957, theorem 5, p 818). Hence, φ is a solution of the equation

$$\mathcal{Q}\varphi(\boldsymbol{w}) + H(\varphi(\boldsymbol{w})) = \mathrm{d}\varphi(\boldsymbol{w})\mathcal{D}\boldsymbol{w}, \qquad \varphi(0) = 0,$$

so that

$$\mathcal{Q}\,\mathrm{d}\varphi(0) = \mathrm{d}\varphi(0)\mathcal{D} \tag{54}$$

follows from a single differentiation and the fact that $H(0) = 0, \mathrm{d}H(0) = 0$. Using the analyticity of φ, let us write

$$\varphi(\boldsymbol{w}) = \mathrm{d}\varphi(0)\boldsymbol{w} + \mathcal{O}(2), \tag{55}$$

where the $\mathcal{O}(2)$ term is a function of $\boldsymbol{w}$. There are now three separate cases to consider regarding the manifolds $\mathcal{M}_p$, $\mathcal{M}_0$ and $\mathcal{M}$.

1. ($\mathcal{M}_p$): The manifold $\mathcal{M}_p$ has the parametric representation in the (ζ, η) plane

$$\mathcal{M}_p = \{(p|\eta|^\rho, \eta)\colon \eta \in \mathbb{R}\} = \{\eta(0, 1) + (p|\eta|^\rho, 0) : \eta \in \mathbb{R}\}$$

and making use of (55), we have

$$\varphi(\mathcal{M}_p) = \{\eta\,\mathrm{d}\varphi(0)[0, 1] + \mathrm{d}\varphi(0)[p|\eta|^\rho, 0] + \mathrm{O}(\eta^2)\colon \eta \text{ small}\},$$

recalling that $\rho > 1$ by assumption (***R***). Hence for $(\lambda, \beta) \in \varphi(\mathcal{M}_p)$ we have

$$\lambda = \eta\theta_1 + \mathrm{o}(\eta), \qquad \beta = \eta\theta_2 + \mathrm{o}(\eta), \tag{56}$$

where $\mathrm{d}\varphi(0)[0, 1] = (\theta_1, \theta_2) \in \mathbb{R}^2$ is a non-zero vector by the invertibility of $\mathrm{d}\varphi(0)$. In (56), $\mathrm{o}(\eta)$ represents terms that are of order $\mathrm{O}(\eta^{\min\{\rho,2\}})$. Using (54), it follows that

$$\begin{aligned}\mathcal{Q}\,\mathrm{d}\varphi(0)[0, 1] &= \mathrm{d}\varphi(0)\mathcal{D}[0, 1]\\ &= \mathrm{d}\varphi(0)[0, \omega - \varsigma] = (\omega - \varsigma)\,\mathrm{d}\varphi(0)[0, 1].\end{aligned}$$

From this we infer that $(\theta_1, \theta_2) \in \langle \boldsymbol{v}\rangle\backslash\{\boldsymbol{0}\}$ so that $\theta_1 \neq 0$ by lemma 2.4.

As a consequence, one can view (56) as defining a $C^{\lfloor\rho\rfloor}$-system of equations which one may solve, using the implicit function theorem, for

$$\beta = \psi_p(\lambda), \qquad \text{such that } \psi_p(0) = 0, \quad \psi_p'(0) = \frac{\theta_2}{\theta_1} = \frac{v_2}{v_1},$$

where ψ_p is $C^{\lfloor\rho\rfloor}$. Using the fact that $b - \mathcal{R}h(b) = \mathcal{Q}\nu = \mathcal{Q}(\lambda, \psi_p(\lambda))$ is satisfied on $W^s(0, 0)$ where h is defined in (50), we can apply the implicit function theorem to write $b = b(\lambda)$ and therefore $a = h(b(\lambda)) =: a(\lambda)$ on $\varphi(\mathcal{M}_p)$.

Now, we can restrict the quasi-linear differential equation (NF) to $\varphi(\mathcal{M}_p)$ in order to obtain a differential equation as follows. From (NF) we have

$$\dot{\alpha} = \dot{\lambda} f_0 + \dot{a} = f_0 + L_0(\lambda f_0 + a) + F(\lambda f_0 + a, \beta)$$

and by taking the inner-product of this with f_0 we obtain

$$\dot{\lambda}\|f_0\|^2 = \|f_0\|^2 + \lambda f_0^{\mathrm{T}} L_0 f_0 + f_0^{\mathrm{T}} F(\lambda f_0 + a(\lambda), \psi_p(\lambda)),$$

on $\varphi(\mathcal{M}_p)$, which we may write as

$$\dot{\lambda} = 1 + \left(\frac{f_0^{\mathrm{T}} L_0 f_0}{\|f_0\|^2}\right)\lambda + F_p(\lambda). \tag{57}$$

It follows that for each p, there is a solution $\lambda_p(\cdot)$ of (57) such that $\lambda_p(0) = 0$ and $\lambda_p \in C^{\lfloor\rho\rfloor+1}$. Since $\alpha_p(t) = \lambda_p(t) f_0 + a(\lambda_p(t))$ and $\beta_p(t) = \psi_p(\lambda(t))$ provides a solution of (NF) for each p, the existence of uncountably many solutions of this quasi-linear problem follows, with α in $C^{\lfloor\rho\rfloor+1}$ and β in $C^{\lfloor\rho\rfloor}$. Since $(x, y) = \chi(\alpha, \beta)$ is a corresponding solution of (1) and (2) from theorem 2.1, we immediately infer that x and y are $C^{\lfloor\rho\rfloor}$, but then (1) tells us that x actually lies in $C^{\lfloor\rho\rfloor+1}$.

2. $(\mathcal{M}_0)$: To cover the case where $p = 0$, reason in a fashion which is entirely analogous to case 1 earlier, except noting that $C^{\lfloor\rho\rfloor}$ can be replaced by C^{ω} throughout.

3. $(\mathcal{M})$: On $\varphi(\mathcal{M})$ we have to invoke the final hypothesis in the statement of the theorem in order to locate the final analytic solution, but the argument is similar to cases 1 and 2. If we recall that $\mathcal{M} = \{(\zeta, \eta)\colon \eta = 0\}$, the manifold $\varphi(\mathcal{M})$ can be represented parametrically (and analytically so) in the form

$$(\lambda, \beta) = \zeta\, \mathrm{d}\varphi(0)[1, 0] + \mathrm{O}(\zeta^2).$$

Now,

$$\begin{aligned}\mathcal{Q}\,\mathrm{d}\varphi(0)[1, 0] &= \mathrm{d}\varphi(0)\mathcal{D}[1, 0]\\ &= \mathrm{d}\varphi(0)[\varsigma, 0] = \varsigma\, \mathrm{d}\varphi(0)[1, 0],\end{aligned}$$

so that $\mathrm{d}\varphi(0)[1, 0] = (\Theta_1, \Theta_2) \in \langle \boldsymbol{u}\rangle\backslash\{0\}$ and if we suppose, seeking a contradiction, that $\mathrm{d}\varphi(0)[1, 0] \in \langle(1, 0)\rangle$ the definition of $\mathcal{Q}$ then yields $\boldsymbol{u} \in \langle(1, 0)\rangle$. However, this in turn gives $\mathcal{Q}(1, 0) = \varsigma(1, 0)$ which contradicts assumption (43) in the statement of the theorem, so that $\Theta_2 \neq 0$. Therefore, $u_2 \neq 0$ also follows.

Therefore, $\varphi(\mathcal{M})$ has the parametric representation

$$\lambda = \zeta\Theta_1 + \mathrm{O}(\zeta^2), \qquad \beta = \zeta\Theta_2 + \mathrm{O}(\zeta^2),$$

which provides the Cartesian, analytic representation of $\varphi(\mathcal{M})$ as a graph

$$\lambda = \psi(\beta), \qquad \text{such that } \psi(0) = 0, \quad \psi'(0) = \frac{\Theta_1}{\Theta_2},$$

obtained using the analytic version of the implicit function theorem.

In order to obtain the restriction of (NF) to $\varphi(\mathcal{M})$, as in case 1 earlier, note that $b - \mathcal{R}h(b) = \mathcal{Q}v = \mathcal{Q}(\psi(\beta), \beta)$ on $\varphi(\mathcal{M})$, and this relationship can be solved near $(b, \beta) = (0, 0)$ for $b = b(\beta)$ by the analytic implicit function theorem. In turn we define $a(\beta) := h(b(\beta))$, using the definition of $W^{\mathrm{s}}(0, 0)$ from (50).

By considering the behaviour of the quasi-linear part of (NF) on $\varphi(\mathcal{M})$, we obtain the analytic, quasi-linear differential equation

$$s(\alpha(\beta), \beta)\dot{\beta} = \psi(\beta)c^{\mathrm{T}} f_0 + \gamma\beta + c^{\mathrm{T}} a(\beta) + G(\alpha(\beta), \beta), \tag{58}$$

where $\alpha(\beta)$ is decomposed as $\psi(\beta) f_0 + a(\beta)$ according to (35). However, we define the function

$$\Omega(\beta) := \frac{\psi(\beta)c^{\mathrm{T}} f_0 + \gamma\beta + c^{\mathrm{T}} a(\beta) + G(\alpha(\beta), \beta}{s(\alpha(\beta), \beta)}$$

and use the fact that $a'(0) = \mathrm{d}h(0,0) \cdot b'(0) = 0$ and $\alpha'(0) = \psi'(0) f_0$, we can apply L'Hôpital's rule and obtain the well-defined, non-zero quantity

$$\Omega(0) = \frac{\psi'(0)c^{\mathrm{T}} f_0 + \gamma}{d_\beta s(0,0) + \psi'(0) d_\alpha s(0,0)[f_0]}. \tag{59}$$

To see that (59) is well-defined and non-zero we reason as follows. Since $\mathcal{Q}(\Theta_1, \Theta_2) = \varsigma(\Theta_1, \Theta_2)$, we see that

$$\frac{\psi'(0)c^{\mathrm{T}} f_0 + \gamma}{d_\beta s(0,0) + \psi'(0) d_\alpha s(0,0)[f_0]} = \frac{\Theta_1 c^{\mathrm{T}} f_0 + \Theta_2 \gamma}{\Theta_2 d_\beta s(0,0) + \Theta_1 d_\alpha s(0,0)[f_0]}$$
$$= \frac{\varsigma \Theta_2}{\varsigma \Theta_1} = \frac{\Theta_2}{\Theta_1} = \frac{u_2}{u_1}.$$

By lemma 2.4 this quantity is well-defined and the fact that $u_2 \neq 0$ was established earlier. It follows that (58) represents an ordinary differential equation near $\beta = 0$, namely $\dot{\beta} = \Omega(\beta)$, and the fact that $\Omega(0) \neq 0$ shows that an analytic solution of this differential equation exists which satisfies $\beta(0) = 0$, $\dot{\beta}(0) \neq 0$. This concludes the proof. □

In case 1 from the proof of theorem 3.1 which describes the behaviour of (NF) along the manifold denoted $\varphi(\mathcal{M}_p)$, it is important to note that the image of $\varphi(\mathcal{M}_p)$ in $U \times \mathbb{R}$, which we denote by W, is transverse to Σ. To see this, use the fact from (56) that $\lambda = \eta\theta_1 + \mathrm{o}(\eta)$ and $\beta = \eta\theta_2 + \mathrm{o}(\eta)$ on $\varphi(\mathcal{M}_p)$, so that $W = \{(\alpha(\eta), \beta(\eta)) : \eta \text{ small}\}$. Elements of this set have the form $\text{const} \cdot \eta(v_1 f_0, v_2) + \mathrm{o}(\eta) \in U \times \mathbb{R}$ and therefore the tangent space of W at zero satisfies $T_0(W) = \langle (v_1 f_0, v_2) \rangle$. Now the tangent space of Σ at zero, from (17) in theorem 2.1, is $T_0(\Sigma) = \{(\alpha \cdot d_\beta s(0,0)[1], -d_\alpha s(0,0)[\alpha]) : \alpha \in U\}$. However, if there is a $(\delta, \mu) \in U \times \mathbb{R}$ such that

$$(\delta \cdot d_\beta s(0,0)[1], -d_\alpha s(0,0)[\delta]) + \mu(v_1 f_0, v_2) = (0,0),$$

it follows that $(d_\beta s(0,0), -d_\alpha s(0,0)[f_0])$ must be an eigenvector for $\mathcal{Q}$. This immediately implies $(\omega - \varsigma) d_\beta s(0,0) = 0$, which contradicts the hypotheses of theorem 3.1 and which therefore ensures that $\dim T_0(\Sigma) \oplus T_0(W) = \dim(U \times \mathbb{R})$. Hence W intersects Σ transversally at zero.

This transversality result is important for reasons which are depicted in figure 3. This shows two invariant manifolds of (41) and (42) through the pseudo-equilibrium, $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ is transverse to Σ but $\mathcal{B}$ is not, and both are stable manifolds. The right-hand diagram of figure 3 shows that due to the change of orientation of the phase-curves where $s(\alpha, \beta) < 0$, the manifold $\mathcal{B}$ cannot be used to form a solution of (NF) as one cannot arrive at, and then leave the singularity along $\mathcal{B}$.

In case the non-resonance condition (***R***) fails in theorem 3.1 we can still prove the following theorem.

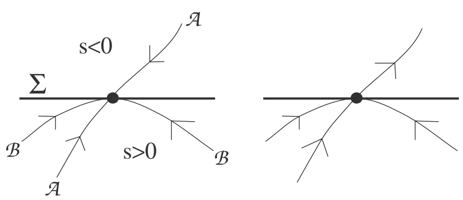

Figure 3. (Left) The flow of the desingularized vector field (41) and (42) along two invariant manifolds ($\mathcal{A}$ and $\mathcal{B}$) associated with a pseudo-equilibrium, with singularity Σ shown as a thick black line. (Right) The orientation of (NF) along $\mathcal{A}$ and $\mathcal{B}$.

Theorem 3.2. *Suppose that* $(0,0)$ *is a proper pseudo-equilibrium point for* (NF)*, that* $\mathcal{Q}$ *has real and distinct eigenvalues* ς *and* $\omega-\varsigma$ *such that* $\mathrm{sgn}(\varsigma)=\mathrm{sgn}(\omega-\varsigma)$ *and*

(R)′ $\rho:=\max\{\varsigma/(\omega-\varsigma),(\omega-\varsigma)/\varsigma\}\in\mathbb{N}\backslash\{0,1\}$,

then there are uncountably many solutions of (NF) (α,β) *in* $C^2\times C^1$*, with initial condition* $(\alpha(0),\beta(0))=(0,0)$*. Consequently, there are uncountably many* $C^2\times C^1$ *solutions* (x,y) *of (1) and (2) with initial condition* $(x(0),y(0))=(0,0)$*.*

Proof. This theorem follows from case 1 of the proof of theorem 3.1 almost verbatim. The only change to note is that the linearizing diffeomorphism φ from that argument is now only of class C^1 (from Hartman (1960) and also see Chiccone and Swanson (2000) for a recent discussion of the regularity of φ), consequently the order relation in (55) must be re-written in the form $\varphi(\boldsymbol{w})=\mathrm{d}\varphi(0)\boldsymbol{w}+\mathrm{o}(\|\boldsymbol{w}\|)$, taking into account the differentiability of φ. □

There is no analogy of theorem 3.1 which covers the case $\rho=1$, because in order to find two linearly independent eigenvectors of $\mathcal{Q}$ corresponding to the eigenvalue ς we must have $\mathcal{Q}=\varsigma I$, where I is the 2×2 identity. However, the latter contradicts the fact that $\mathcal{Q}_{12}\neq0$ which itself follows from (A3). If $\mathcal{Q}$ has an algebraically double eigenvalue ς with geometric multiplicity 1, then logarithmic terms ensure that the invariant manifolds of the linear problem $\omega b'=\mathcal{Q}b$ from (49) are insufficiently smooth to allow one to find multiple solutions of (NF).

The following example shows that if $\mathcal{Q}$ has an eigenvalue of algebraic multiplicity 2 then (NF) need not support more than one solution through the point $(\alpha,\beta)=(0,0)$, and in this example $\mathcal{Q}$ is the matrix

$$\begin{pmatrix}a & 1\\ 0 & a\end{pmatrix}.$$

Example 1. *Consider the quasi-linear differential equation*

$$\dot{x}=1,\qquad (ax+y)\dot{y}=ay, \tag{60}$$

where $a\in\mathbb{R}$ *is non-zero and suppose that (60) has a solution* $(x(t),y(t))$ *on some interval of* $t=0$ *with initial condition* $(x(0),y(0))=(0,0)$*. Clearly* $(x(t),y(t))=(t,0)$ *is one solution, and if* $y(t)\not\equiv0$ *then we can integrate (60) to obtain the relationship*

$$x(t)=t,\qquad y(t)=\bar{A}\exp\left(\frac{t}{ay(t)}\right)\quad(\bar{A}\in\mathbb{R}).$$

As $y(t)$ *is a differentiable function then* $y(0)\neq0$ *follows, so no such solution exists.*

3.2. Pseudo-saddles

We now present the second theorem regarding singular solutions of (1) and (2), a pictorial representation of which is given in figure 4.

Theorem 3.3. *Suppose that* $(0,0)$ *is a proper pseudo-equilibrium point for* (NF)*, that* $\mathcal{Q}$ *has real eigenvalues* $\varsigma,\omega-\varsigma$ *and*

$$\mathrm{sgn}(\varsigma)\neq\mathrm{sgn}(\omega-\varsigma),$$

then there are exactly two analytic solutions (α,β) *of* (NF) *with initial condition* $(\alpha(0),\beta(0))=(0,0)$*. Consequently, there are exactly two analytic solutions* (x,y) *of (1) and (2) with initial condition* $(x(0),y(0))=(0,0)$*.*

In addition, there are two piecewise analytic solutions of (NF)*,* $(\alpha,\beta)\in C^1(I)\times W^{1,\infty}(I)$*, such that* $(\alpha(0),\beta(0))=(0,0)$*. Consequently, there are exactly two piecewise analytic solutions of (1) and (2) such that* $(x,y)\in C^1(I)\times C^0(I)$ *and* $k^{\mathrm{T}}y\in W^{1,\infty}(I)$*, with initial condition* $(x(0),y(0))=(0,0)$*.*

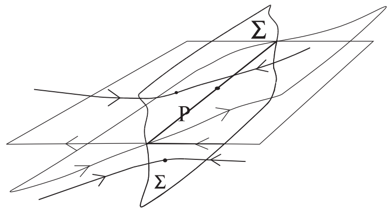

Figure 4. The flow of a DAE near a pseudo-saddle. There are two analytic and two solutions of finite smoothness which pass through the singularity. Singular points not on P are impasse points.

Proof. The proof follows that of theorem 3.1 verbatim up to equation (49), so let us take the notation as used in that proof and now proceed in the following manner.

By assumption, $\mathcal{Q}$ has a positive and a negative eigenvalue, so that the differential equation

$$\begin{bmatrix} b' \\ a' \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} + \mathcal{O}(2) \tag{61}$$

has stable, unstable and centre manifolds associated with its zero equilibrium; let us consider the stable (W^{s}) and unstable (W^{u}) manifolds, both of which are analytic.

As in theorem 3.1, again define the matrix $\mathcal{P} = [\boldsymbol{u}|\boldsymbol{v}]$ so that $\mathcal{QP} = \mathcal{PD}$, where $\mathcal{D}$ is the diagonal matrix of eigenvalues $\mathrm{diag}\{\varsigma, \omega - \varsigma\}$. On setting $\mathcal{P}\boldsymbol{w} = b$ with $\boldsymbol{w} = (\zeta, \eta)$ we obtain

$$\omega \boldsymbol{w}' = \mathcal{D}\boldsymbol{w}, \qquad \omega a' = H(a, \boldsymbol{w}), \tag{62}$$

where $H(0, 0) = \mathbf{0}$ and $\mathrm{d}H(0, 0) = 0$ and H represents the $\mathcal{O}(2)$ terms in (61).

Now, assuming without the loss of any generality (reversing time if need be) that $\varsigma > 0, \omega - \varsigma < 0$, we can describe W^{s} and W^{u} in terms of the coordinates given in (62). We therefore obtain a description of

$$W^{\mathrm{s}} \qquad \text{as } \zeta = \zeta_S(\eta), \quad a = a_S(\eta), \tag{63}$$

such that $\zeta_S(0) = 0$, $\mathrm{d}\zeta_S(0) = 0$, $a_S(0) = 0$, $\mathrm{d}a_S(0) = 0$; and

$$W^{\mathrm{u}} \qquad \text{as } \eta = \eta_U(\zeta), \quad a = a_U(\zeta), \tag{64}$$

such that $\eta_U(0) = 0$, $\mathrm{d}\eta_U(0) = 0$, $a_U(0) = 0$ and $\mathrm{d}a_U(0) = 0$.

Let us recall that

$$b = \mathcal{Q}\nu + \mathcal{R}a,$$

where $\nu = (\lambda, \beta)$, λ is defined in (35) and $\mathcal{R}$ in (47). Consequently, on W^{s} we have

$$\mathcal{P}\begin{pmatrix} \zeta_S(\eta) \\ \eta \end{pmatrix} = \mathcal{Q}\begin{pmatrix} \lambda \\ \beta \end{pmatrix} + \mathcal{R}a_S(\eta),$$

which we may write as

$$\eta\mathcal{P}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathcal{Q}\begin{pmatrix} \lambda \\ \beta \end{pmatrix} + \mathrm{O}(\eta^2). \tag{65}$$

However, by definition $\mathcal{P}(0, 1) = \boldsymbol{v} = (v_1, v_2)$, so that from (65) and the fact that $\mathcal{Q}^{-1}\boldsymbol{v} = (\omega - \varsigma)^{-1}\boldsymbol{v}$, we obtain

$$(\lambda, \beta) = \eta(\omega - \varsigma)^{-1}(v_1, v_2) + \mathrm{O}(\eta^2), \tag{66}$$

as a parametric description of W^{s} in the (λ, β) plane. Similarly, on W^{u} we have

$$\mathcal{P}\begin{pmatrix} \zeta \\ \eta_U(\zeta) \end{pmatrix} = \mathcal{Q}\begin{pmatrix} \lambda \\ \beta \end{pmatrix} + \mathcal{R}a_U(\zeta),$$

which we may write as

$$\zeta \mathcal{P}\begin{pmatrix}1\\0\end{pmatrix} = \mathcal{Q}\begin{pmatrix}\lambda\\\beta\end{pmatrix} + \mathrm{O}(\zeta^2).$$

Since $\mathcal{P}(1,0) = \boldsymbol{u} = (u_1, u_2)$, we obtain

$$(\lambda, \beta) = \zeta\varsigma^{-1}(u_1, u_2) + \mathrm{O}(\zeta^2), \tag{67}$$

as a parametric description of W^{u} in the (λ, β) plane.

Recall that $v_1 \cdot u_1 \neq 0$ from lemma 2.4, this will be used below. One can represent W^{s} as a graph by viewing (66) as an equation which first allows one to solve the relationship $\lambda = \eta(\omega - \varsigma)^{-1}v_1 + \mathrm{O}(\eta^2)$ near $(\lambda, \eta) = (0, 0)$ using the implicit function theorem as

$$\eta = (\omega - \varsigma)v_1^{-1}\lambda + \mathrm{O}(\lambda^2) =: \eta_S(\lambda)$$

and then

$$\beta = v_2 v_1^{-1}\lambda + \mathrm{O}(\lambda^2) \qquad \text{on } W^{\mathrm{s}} \tag{68}$$

follows from (66).

One may similarly represent W^{u} as a graph by viewing (67) as an equation whereby one solves the relationship $\lambda = \zeta\varsigma^{-1}u_1 + \mathrm{O}(\zeta^2)$ near $(\lambda, \zeta) = (0, 0)$ as

$$\zeta = \varsigma u_1^{-1}\lambda + \mathrm{O}(\lambda^2) =: \zeta_U(\lambda)$$

using the implicit function theorem, giving

$$\beta = u_2 u_1^{-1}\lambda + \mathrm{O}(\lambda^2) \qquad \text{on } W^{\mathrm{u}}. \tag{69}$$

Now we consider (NF) restricted to the graphs given in (68) and (69). Since $\alpha = \lambda f_0 + a$ with $a^{\mathrm{T}} f_0 = 0$, let us project the ordinary part of (NF) orthogonally onto f_0 to provide the differential equation

$$\dot\lambda = 1 + \left(\frac{f_0^{\mathrm{T}} L_0 f_0}{\|f_0\|^2}\right)\lambda + \frac{f_0^{\mathrm{T}} L_0 a}{\|f_0\|^2} + f_0^{\mathrm{T}} F(\lambda f_0 + a, \beta). \tag{70}$$

On W^{s}, a is given by the expression $a_S(\eta) = a_S((\omega - \varsigma)v_1^{-1}\lambda + \mathrm{O}(\lambda^2)) = \mathrm{O}(\lambda^2)$ (where the second-order property here follows from (63) and the vanishing of a_S and its derivative at the origin) and β is given by $v_2 v_1^{-1}\lambda + \mathrm{O}(\lambda^2)$, thus (70) can be written

$$\dot\lambda = 1 + \left(\frac{f_0^{\mathrm{T}} L_0 f_0}{\|f_0\|^2}\right)\lambda + \mathrm{O}(\lambda^2). \tag{71}$$

This is an analytic differential equation which has exactly one solution which satisfies $\lambda(0) = 0$, thus providing one of the analytic solutions in the statement of the theorem.

On W^{u}, using (64), a is given by the expression $a_U(\zeta) = a_U(\varsigma u_1^{-1}\lambda + \mathrm{O}(\lambda^2)) = \mathrm{O}(\lambda^2)$ and β is given by $u_2 u_1^{-1}\lambda + \mathrm{O}(\lambda^2)$, and the same reasoning as that used to derive equation (71) shows that a second analytic solution can be found.

This concludes the proof of the existence of two analytic solutions which we shall label as $(\lambda^{\mathrm{s}}, \beta^{\mathrm{s}})$ and $(\lambda^{\mathrm{u}}, \beta^{\mathrm{u}})$, both of which satisfy $(\lambda(0), \beta(0)) = (0, 0)$. Note that $\dot\lambda^{\mathrm{s}}(0) = \dot\lambda^{\mathrm{u}}(0) = 1$ whereas $\dot\beta^{\mathrm{s}}(0) = v_2/v_1$ and $\dot\beta^{\mathrm{u}}(0) = u_2/u_1$, moreover the fact that $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent ensures that $\dot\beta^{\mathrm{s}}(0) \neq \dot\beta^{\mathrm{u}}(0)$.

In order to construct the two solutions of reduced regularity given in the statement of the theorem, we proceed by concatenating the two analytic solutions in the following manner. Let $(-T, T)$ be a common interval of existence of $(\lambda^{\mathrm{s}}, \beta^{\mathrm{s}})$ and $(\lambda^{\mathrm{u}}, \beta^{\mathrm{u}})$ and define the functions

$$L^{\mathrm{u}}(t) := \begin{cases}\lambda^{\mathrm{u}}(t): & t \in (-T, 0),\\ \lambda^{\mathrm{s}}(t): & t \in [0, T),\end{cases} \qquad B^{\mathrm{u}}(t) := \begin{cases}\beta^{\mathrm{u}}(t): & t \in (-T, 0),\\ \beta^{\mathrm{s}}(t): & t \in [0, T)\end{cases}$$

and

$$L^{\mathrm{s}}(t) := \begin{cases} \lambda^{\mathrm{s}}(t)\colon & t \in (-T, 0), \\ \lambda^{\mathrm{u}}(t)\colon & t \in [0, T), \end{cases} \qquad B^{\mathrm{s}}(t) := \begin{cases} \beta^{\mathrm{s}}(t)\colon & t \in (-T, 0), \\ \beta^{\mathrm{u}}(t)\colon & t \in [0, T). \end{cases}$$

Hence $\dot{L}^{\mathrm{u}}(0) = 1$ and $\dot{L}^{\mathrm{s}}(0) = 1$, so that L^{u} and L^{s} are both in $C^1(-T, T)$ as they are both analytic on $(-T, T)\backslash\{0\}$. We also define the functions

$$A^{\mathrm{s}}(t) := \begin{cases} a_S(\eta_S(\lambda^{\mathrm{s}}(t)))\colon & t \in (-T, 0), \\ a_U(\zeta_U(\lambda^{\mathrm{u}}(t)))\colon & t \in [0, T) \end{cases}$$

and

$$A^{\mathrm{u}}(t) := \begin{cases} a_U(\zeta_U(\lambda^{\mathrm{u}}(t)))\colon & t \in (-T, 0), \\ a_S(\eta_S(\lambda^{\mathrm{s}}(t)))\colon & t \in [0, T) \end{cases}$$

and check that

$$(\alpha^{\mathrm{u}}(t), \beta^{\mathrm{u}}(t)) := (L^{\mathrm{u}}(t) f_0 + A^{\mathrm{u}}(t), B^{\mathrm{u}}(t))$$

and

$$(\alpha^{\mathrm{s}}(t), \beta^{\mathrm{s}}(t)) := (L^{\mathrm{s}}(t) f_0 + A^{\mathrm{s}}(t), B^{\mathrm{s}}(t))$$

form solutions of (NF) on $(-T, T)$. Let us now do this. The given pair $(\alpha^{\mathrm{u}}, \beta^{\mathrm{u}})$ is a solution of (NF) on $(-T, 0)$ and $(0, T)$ separately, and

$$\alpha^{\mathrm{u}}(0) = 0, \qquad \dot{\alpha}^{\mathrm{u}}(0) = f_0, \qquad \alpha^{\mathrm{u}} \in C^1(-T, T)$$

and

$$\beta^{\mathrm{u}}(0) = 0, \qquad \beta^{\mathrm{u}} \in W^{1,\infty}(-T, T)$$

are all true by construction. Since $s(\alpha^{\mathrm{u}}(0), \beta^{\mathrm{u}}(0)) = s(0, 0) = 0$, it follows that we can assume (by a suitable modification on a set of measure zero) that the function $s(\alpha^{\mathrm{u}}, \beta^{\mathrm{u}})\dot{\beta}^{\mathrm{u}}$, which is an element of $L^\infty(-T, T)$, is actually in $C^0(-T, T)$. It is now immediate that $(\alpha^{\mathrm{u}}, \beta^{\mathrm{u}})$ is a solution of (NF), and the same reasoning applies to $(\alpha^{\mathrm{s}}, \beta^{\mathrm{s}})$. □

3.3. Pseudo-centres

This is the final case that we consider, whereby $\det(\mathcal{Q}) = 0$ so that lemma 2.3 is not applicable and the set of pseudo-equilibria of (NF), P, does not necessarily form a manifold.

Theorem 3.4. *Suppose that* $(0, 0)$ *is a proper pseudo-equilibrium point for* (NF) *and* $\mathcal{Q}$ *has real and distinct eigenvalues* $0, \omega$ *(and* $\omega \neq 0$ *by (A2)). Then there is an analytic solution* (α, β) *of* (NF) *with initial condition* $(\alpha(0), \beta(0)) = (0, 0)$ *and consequently there is an analytic solution* (x, y) *of (1) and (2) with initial condition* $(x(0), y(0)) = (0, 0)$.

Moreover, for each integer $r \geqslant 1$ *there is a codimension-1, quasi-invariant,* C^r *manifold,* $W^{\mathrm{R}} \subset U \times \mathbb{R}$*, of* (NF)*. So, for each* $(\alpha_0, \beta_0) \in W^{\mathrm{R}}$ *there is a solution of* (NF) *in* W^{R} *with* $(\alpha, \beta) \in C^{r+1}(I) \times C^r(I)$*, for some interval* I *containing* 0*, and such that* $(a(0), \beta(0)) = (\alpha_0, \beta_0)$*. This yields a solution* (x, y) *of (1) and (2) with initial condition in* $\chi(W^{\mathrm{R}})$*, where* χ *is defined in theorem 2.1, such that* $(x, y) \in C^{r+1}(I) \times C^r(I)$*. Finally,* $\hat{P} := W^{\mathrm{R}} \cap \Sigma$ *is a set of proper pseudo-equilibria of* (NF) *containing the point* $(\alpha, \beta) = (0, 0)$.

Proof. The proof follows that of theorem 3.1 verbatim up to equation (47), but now $\det(\mathcal{Q}) = 0$. Let us therefore assume that

$$\mathcal{Q}\boldsymbol{u} = \boldsymbol{0} \qquad \text{and} \qquad \mathcal{Q}\boldsymbol{v} = \omega\boldsymbol{v},$$

so that $\varsigma = 0$, and since $\omega \neq 0$ it follows that $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent. Recalling the definition of the matrix $\mathcal{P} = [\boldsymbol{u}|\boldsymbol{v}]$, set $\mathcal{P}\boldsymbol{w} = b$ with $\boldsymbol{w} = (\zeta, \eta)$, so that $\zeta\boldsymbol{u} + \eta\boldsymbol{v} = \boldsymbol{w}$ and (46) can be written

$$\zeta' = \omega^{-1} P_u[\mathcal{R}a] + \mathcal{O}(2), \tag{72}$$

$$\eta' = \eta + \omega^{-1} P_v[\mathcal{R}a] + \mathcal{O}(2), \tag{73}$$

$$a' = \mathcal{O}(2), \tag{74}$$

where the projection mappings P_u and P_v project $\langle \boldsymbol{u}, \boldsymbol{v}\rangle$ onto its subspaces $\langle \boldsymbol{u}\rangle$ and $\langle \boldsymbol{v}\rangle$, respectively. If we now define $\bar{\eta} = \eta + (1/\omega)P_u[\mathcal{R}a]$ then $\bar{\eta}' = \bar{\eta} + \mathcal{O}(2)$, and (72)–(74) can be written

$$\begin{pmatrix} \bar{\eta}' \\ \zeta' \\ a' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^{-1} P_v[\mathcal{R}] \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\eta} \\ \zeta \\ a \end{pmatrix} + \mathcal{O}(2). \tag{75}$$

Let $r \geqslant 1$ be given. Now, (75) has a one-dimensional unstable manifold (W^{u}) and a C^r, $(n-1)$-dimensional centre manifold (W^{c}) associated with the equilibrium point $(\bar{\eta}, \zeta, a) = (0, 0, 0) \in \mathbb{R} \times \mathbb{R} \times \langle f_0\rangle^{\perp}$ (where $\langle f_0\rangle^{\perp}$ is a codimension-1 subset of U, which itself has dimension $n-1$), both of which are given by graphs such that

$$\text{on } W^{\mathrm{u}}: \qquad \zeta = \zeta_U(\bar{\eta}), \qquad a = a_U(\bar{\eta})$$

and

$$\text{on } W^{\mathrm{c}}: \qquad \bar{\eta} = \bar{\eta}_C(a, \zeta).$$

Let us also record the following properties: $\zeta_U(0) = 0$, $a_U(0) = 0$, $\mathrm{d}\zeta_U(0) = 0$, $\mathrm{d}a_U(0) = 0$ and both ζ_U and a_U are analytic; $\bar{\eta}_C(0, 0) = 0$, $\mathrm{d}\bar{\eta}_C(0, 0) = 0$ and for any given $r \in \mathbb{N}$, $\bar{\eta}_C$ is a C^r function on some neighbourhood $B_\delta(0, 0)$ of $(0, 0)$ in $\langle f_0\rangle^{\perp} \times \mathbb{R}$, where this neighbourhood may depend on r.

Consider the behaviour of (NF) restricted to W^{u}. The relation $\eta = \bar{\eta} - \omega^{-1} P_u[\mathcal{R}a_U(\bar{\eta})]$ inverts near zero to give $\bar{\eta} = \eta + \mathrm{O}(\eta^2)$, so that

$$(\lambda, \beta) = \eta\boldsymbol{u} + \zeta_U(\eta + \mathrm{O}(\eta^2))\boldsymbol{v}$$

is a parametric representation of W^{u} in the (λ, β)-plane. Since $\boldsymbol{u} = (u_1, u_2)$ with $u_1 \neq 0$, we can eliminate η from this representation to give the Cartesian representation $\beta = \lambda u_2/u_1 + \mathrm{O}(\lambda^2)$ locally to zero. Having expressed the manifold W^{u} locally as a graph over the variable λ, we may project the ordinary part of (NF) orthogonally onto the span of f_0 and restrict (NF) to W^{u}. This provides an analytic differential equation on W^{u} of the form $\dot{\lambda} = 1 + \mathrm{O}(\lambda)$, thus providing an analytic solution of (NF) through the point $(\alpha, \beta) = (0, 0)$.

On W^{c} we have $\eta = \bar{\eta}_C(a, \zeta) - \omega^{-1} P_u[\mathcal{R}a] =: \eta(a, \zeta)$, and it follows that

$$\begin{pmatrix} \lambda \\ \beta \end{pmatrix} = \eta(a, \zeta)\boldsymbol{u} + \zeta\boldsymbol{v}. \tag{76}$$

However, relationship (76) can be locally solved using the implicit function theorem to give $\eta = \eta_C(a, \lambda)$ and $\beta = \beta_C(a, \lambda)$. Using the fact that $\alpha = \lambda f_0 + a$ we simply write $\beta = \beta_C(a)$ for this representation and from the ordinary part of (NF) we immediately obtain a differential equation on W^{c}, namely

$$\dot{\alpha} = f_0 + L_0\alpha + F(\alpha, \beta_C(\alpha)).$$

(Since (NF) thus induces a local flow on W^{c}, we re-write W^{R} as W^{c}: a superscript c would be inappropriate and we use a superscript R to remind us of the regular character of solutions of (NF) on W^{c}.)

Now consider the non-empty set

$$\hat{P} := \{(\alpha, \beta) \in B'' : s(\alpha, \beta) = 0, \beta = \beta_C(\alpha), \alpha = \lambda f_0 + a\} \subset \Sigma \cap W^{\mathrm{R}},$$

where $B'' \subset B'$ is some open neighbourhood of zero and B' is defined in theorem 2.1. We now show that B'' can be chosen so that $\hat{P}$ is a set of pseudo-equilibria for (NF) according to definition 1, recalling that $f_1 = 0$ by (A5). So let $(\alpha_0, \beta_0) \in \hat{P}$ and let (α, β) be a solution of (41) and (42) with (α_0, β_0) as initial condition such that $(\alpha(\tau), \beta(\tau)) \in W^{\mathrm{R}}$ for all τ sufficiently small, then $\beta(\tau) \equiv \beta_C(\alpha(\tau))$, where the mapping β_C is defined above. For all such τ there results

$$\begin{aligned} c^{\mathrm{T}}\alpha + \gamma\beta + G(\alpha, \beta) &= \beta' \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\beta_C(\alpha(\tau)) \\ &= \mathrm{d}\beta_C(\alpha)\alpha'(\tau) \\ &= \mathrm{d}\beta_C(\alpha)s(\alpha, \beta)(f_0 + L_0\alpha + F(\alpha, \beta_C(\alpha)), \end{aligned}$$

setting $\tau = 0$ yields $c^{\mathrm{T}}\alpha_0 + \gamma\beta_0 + G(\alpha_0, \beta_0) = 0$. Now $f_0 \neq 0$ as $f(0,0) = f_0 + f_1 Bk$ and both $f_1 = 0$ and (A5) are satisfied, so we choose B'' so that $f_0 + L_0\alpha + F(\alpha, \beta_C(\alpha)) \neq 0$ for all $(\alpha, \beta) \in B''$. From this it follows that $\hat{P}$ forms a set of proper pseudo-equilibria as claimed. □

Theorem 3.4 provides another case, distinct from the singular equilibrium problem described in theorem 2.5 of Beardmore and Laister (2002), where (NF) has a manifold of smooth solutions which has non-empty intersection with the singularity, and therefore a local flow is induced on that manifold by (NF). Figure 5 shows the structure of the invariant manifolds near such a proper pseudo-centre.

We conclude this paper with an extended example to illustrate the role played by the matrix $\mathcal{Q}$ in the preceding theorem.

Example 2. *Consider the following equation*

$$\tfrac{1}{2}(\dot{\theta})^2 + (1 - \mathrm{e}^{-t})F(\theta) + \mathrm{e}^{-t}\theta^2 = \lambda, \tag{77}$$

where λ *is a given parameter and* $F(\theta)$ *is an analytic function. We seek solutions of (77) which satisfy the initial conditions* $\theta(0) = 0, \dot{\theta}(0) = 0$*. Now (77) can be written as a DAE by setting* $\Omega = \dot{\theta}$*. This gives a system of the form*

$$\begin{aligned} \dot{\theta} &= \Omega, \\ \dot{s} &= 1 - s, \\ 0 &= \tfrac{1}{2}\Omega^2 + (1 - s)F(\theta) + s\theta^2, \end{aligned}$$

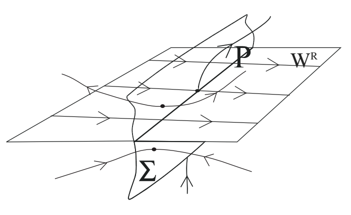

Figure 5. The structure of invariant manifolds near a pseudo-centre. A dynamical system is induced by the DAE on W^{R}, $W^{\mathrm{R}} \cap P$ are also pseudo-equilibria and singular points not on P are impasse points. There is also an analytic solution through the pseudo-centre which does not lie within W^{R}.

which we write as

$$\begin{pmatrix}\dot{s}\\ \dot{\theta}\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ -\lambda & \vartheta & 0\end{pmatrix}\begin{pmatrix}s\\ \theta\\ \Omega\end{pmatrix} + \begin{pmatrix}0\\ 0\\ \mathcal{O}(2)\end{pmatrix}, \tag{78}$$

in the vicinity of the point $(s, \theta; \Omega) = (0, 0; 0)$*, where* $\vartheta = F'(0)$ *and* $\lambda = F(0)$*. In terms of the notation used thus far in this paper, we have* $x = (s, \theta)$, $y = \Omega$*, and* $f(x, y) = f(s, \theta; \Omega) = (1 - s, \Omega)$, $g(x, y) = g(s, \theta; \Omega) = -\lambda + \frac{1}{2}\Omega^2 + (1 - s)F(\theta) + s\theta^2$.

It follows that $g(0, 0; 0) = 0$ *and* $f(0, 0; 0) = (1, 0) \neq (0, 0)$*, so that* $(s, \theta; \Omega) = (0, 0; 0)$ *is a candidate proper pseudo-equilibrium point, as* $d_y g(0, 0)[k]$ *from (A1) corresponds to* $d_\Omega g(0, 0; 0) = 0$*, with* $u = k = 1$*. Condition (A1) of definition 2 is therefore seen to be satisfied. Now,* $d_x g(0, 0)$ *corresponds to the vector*

$$d_{(s,\theta)} g(0, 0; 0) = (d_s g(0, 0; 0), d_\theta g(0, 0; 0)) = (-\lambda, \vartheta),$$

and $d_y f(0, 0)$ *corresponds to* $d_\Omega f(0, 0; 0) = (0, 1)$*. Consequently,* $u^{\mathrm{T}} d_x g(0, 0) d_y f(0, 0)k = (-\lambda, \vartheta) \cdot (0, 1) = \vartheta$ *where a dot* $(\cdot)$ *is the usual inner product on* $\mathbb{R}^2$*. In order to satisfy condition (A2) of definition 2 we need* $F'(0) \neq 0$ *to hold. Condition (A3) of definition 2 is clearly satisfied since* $d^2_{yy} g(0, 0)[k, k]$ *corresponds to the expression* $d^2_{\Omega\Omega} g(0, 0; 0) = 1 \neq 0$*. The mapping* $\mathrm{d}(f \times g)$ *in condition (A4) is given by the matrix in (78):*

$$L := \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ -\lambda & \vartheta & 0\end{pmatrix}$$

and this is invertible by the fact that $\vartheta \neq 0$.

Consequently, $(s, \theta; \Omega) = (0, 0; 0)$ *is a folded-singular point when* $\lambda = 0$ *and if is not to be an impasse point for (77), then we must also ensure that (A5) holds. For this to be the case, we need the expression corresponding to* $d_x g(0, 0) f(0, 0)$ *from (A5), namely* $(-\lambda, F'(0)) \cdot f(0, 0; 0) = (-\lambda, \vartheta) \cdot (1, 0) = -\lambda$*, to be zero. Thus we must also assume that* $F(0) = 0$ *to ensure that the given singular point is also a proper pseudo-equilibrium point.*

Let us now determine the matrix $\mathcal{Q}$ *in order to determine the type of pseudo-equilibrium point. The vectors denoted* Bk *and* $C^{\mathrm{T}}u$ *in lemma 2.1 are*

$$Bk = (0, 1)^{\mathrm{T}} \qquad \text{and} \qquad C^{\mathrm{T}}u = (0, \vartheta),$$

this can be read from the definition of the matrix L above which is partitioned according to (9). Hence $\omega = u^{\mathrm{T}} C B k = \vartheta$*, and the space* $U = \langle C^{\mathrm{T}}u\rangle^{\perp} (\subset \mathbb{R}^2)$ *is given by* $U = \langle(0, \vartheta)\rangle^{\perp} = \langle(1, 0)\rangle$*. We may assume without the loss of any generality (by rescaling time if necessary) that*

$$\vartheta = 1 \tag{79}$$

and consequently the space $U \oplus \langle Bk\rangle$ *from (13), with associated complementary projections* P_U *and* P_B*, has orthonormal basis* $\{(0, 1), (1, 0)\}$.

Now, the vector f_0 *from theorem 2.1 is given by*

$$f_0 = P_U f(0, 0) = P_U(1, 0) = (1, 0),$$

and $f_1 = P_B f(0, 0)$ *from the same theorem is* $f_1 = (1, 0) \cdot (0, 1) = 0$*. (The latter is simply a repetition of the fact that we have a pseudo-equilibrium point at* $(s, \theta; \Omega) = (0, 0; 0)$*.)*

Now $d^2_{xy} g(0, 0)$ *is the mapping*

$$d^2_{(s,\theta)\Omega} g(0, 0; 0) = (d^2_{s\Omega} g(0, 0; 0), d^2_{\theta\Omega} g(0, 0; 0)) = (0, 0)$$

and $d^2_{yy}g(0,0)$ *is* $d^2_{\Omega\Omega}g(0,0;0) = 1$. *Also,* $d^2_{xx}g(0,0)$ *corresponds to*

$$\begin{pmatrix} d^2_{ss}g(0,0;0) & d^2_{\theta s}g(0,0;0) \\ d^2_{s\theta}g(0,0;0) & d^2_{\theta\theta}g(0,0;0) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & F''(0) \end{pmatrix}.$$

Now we are in a position to write down the matrix $\mathcal{Q}$*:*

$$\begin{aligned} \mathcal{Q}_{11} &= -u^{\mathrm{T}} d^2_{xy}g(0,0)[f_0, k] = (0,0)\cdot(1,0) = 0, \\ \mathcal{Q}_{12} &= -u^{\mathrm{T}} d^2_{yy}g(0,0)[k, k] = -1, \\ \mathcal{Q}_{22} &= \omega + u^{\mathrm{T}} d^2_{xy}g(0,0)[f_0, k] = \omega = \vartheta = 1. \end{aligned}$$

In order to find $\mathcal{Q}_{21} = -\omega k^{\mathrm{T}} C_1 A_0^{-1} f_0 + u^{\mathrm{T}} d^2_{xx}g(0,0)[f_0, f_0]$*, let us note that the expression* $u^{\mathrm{T}} d^2_{xx}g(0,0)[f_0, f_0]$ *corresponds to*

$$f_0 \cdot \begin{pmatrix} 0 & -1 \\ -1 & F''(0) \end{pmatrix} (1,0)^{\mathrm{T}} = (1,0)\cdot(0,-1) = 0,$$

so that $\mathcal{Q}_{21} = -\omega k^{\mathrm{T}} C_1 A_0^{-1} f_0$*, where* C_1 *is defined in (10) and* A_0 *is defined in (25) and* $A_0^{-1} = L_0$ *in terms of theorem 2.1.*

In order to determine these final elements of $\mathcal{Q}_{21}$*, note that*

$$L^{-1} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = L,$$

since $\lambda = 0$ *and* $\vartheta = 1$. *It follows that* $C_1 = (0,1)$ *and* A_0 *is the mapping* A_1 *restricted to the invariant subspace* $\langle(1,0)\rangle$*, so* $A_0(\mu, 0) = (-\mu, 0)$ *for all* $\mu \in \mathbb{R}$. *Hence* $A_0^{-1}(\mu, 0) = (-\mu, 0)$ *for all* $\mu \in \mathbb{R}$ *too and therefore* $\mathcal{Q}_{21} = -(0,1)A_0^{-1}(1,0) = -(0,1)\cdot(-1,0) = 0$. *Finally*

$$\mathcal{Q} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

and the pseudo-equilibrium point of (77) is a pseudo-centre since $\sigma(\mathcal{Q}) = \{0, 1\}$. *Theorem 3.4 now applies and the existence of two solutions through the pseudo-equilibrium point follows.*

In order to see a little more directly how the matrix $\mathcal{Q}$ *arises in this simple case, note that the constraint (78) can be solved near to the point* $(s, \theta; \Omega) = (0, 0; 0)$ *for*

$$\theta = -\tfrac{1}{2}\Omega^2 + p(s, \Omega^2),$$

by the implicit function theorem because $F'(0) = 1$ *was assumed in (79), where* p *represents terms of order three and higher. Differentiating (78) we obtain*

$$\Omega\dot{\Omega} + F'(\theta)\dot{\theta}(1-s) - \dot{s}F(\theta) + \dot{s}\theta^2 + 2s\theta\dot{\theta} = 0,$$

from where

$$\Omega\dot{\Omega} + F'(\theta)\Omega(1-s) - (1-s)F(\theta) + (1-s)\theta^2 + s\theta\Omega = 0.$$

However, $F(\theta) = \theta + \mathrm{O}(\theta^2)$ *which is* $\mathcal{O}(2)$ *as a function of* (Ω, s)*, and we subsequently obtain a relationship of the form*

$$\dot{s} = 1 - s, \qquad \Omega\dot{\Omega} + \Omega = \mathcal{O}(2), \tag{80}$$

which is satisfied by sufficiently smooth solutions of (77); recall that θ *is a function of* s *and* Ω^2. *Here* $\mathcal{O}(2)$ *denotes a function of* Ω *and* s *of order 2.*

Rescaling time in (80) we obtain a differential equation whose orbits coincide with those of (80) away from the singular manifold $\{(s,\Omega) : \Omega = 0\}$. *One choice for this rescaling yields*

$$s' = -\Omega + s\Omega, \qquad \Omega' = \Omega + \mathcal{O}(2), \tag{81}$$

the linearization of which about the point $(s, \Omega) = (0, 0)$ *is the matrix* $\mathcal{Q}$*. The invariant manifolds of (81) can now be obtained; there is a centre manifold and an unstable manifold and the corresponding graphs can now be used in (80) to show that this quasi-linear problem has smooth solutions as described in theorem 3.4.*

3.4. Complex eigenvalues

One situation not covered previously is when the matrix $\mathcal{Q}$ has complex eigenvalues, in this case we have the following.

Theorem 3.5. *Suppose that* $\sigma(\mathcal{Q}) = \{u + \mathrm{i}v, u - \mathrm{i}v\}$ *where* $u, v \in \mathbb{R}$ *and* $u, v \neq 0$*, then there is no solution of* (NF) *with initial condition* $(\alpha, \beta) = (0, 0)$.

This theorem is a direct consequence of the fact that the focus studied in the planar singular systems of Llibre *et al* (2002) (this reference is to be found in Galves *et al* (2002)) can support no smooth solution. This follows from the geometric fact that any orbit emanating from such a focus must intersect the codimension-1 singularity Σ infinitely many times. When such intersection points are impasse points, as happens in the case of (NF) by the fact that points on $\Sigma \backslash P$ are impasse points, we then infer that any interval of existence of a solution through such a focus must contain infinitely many points where the corresponding orbit encounters impasse points. This can only lead to the conclusion that the interval of existence is a singleton set, so no such solution exists.

It is easy to see directly that a two-dimensional quasi-linear problem like (NF) for which the matrix $\mathcal{Q}$ has complex eigenvalues exhibits no solutions in the following way. If there is a differentiable solution of

$$\dot{x} = 1 + \mathcal{O}(2), \qquad (\bar{a}x + \bar{b}y + \mathcal{O}(2))\dot{y} = \bar{c}x + \bar{d}y + \mathcal{O}(2)$$

subject to the initial condition $(x(0), y(0)) = (0, 0)$, then

$$(\bar{a} + \bar{b}\dot{y}(0))\dot{y}(0) = \bar{c} + \bar{d}\dot{y}(0). \tag{82}$$

The fact that

$$\mathcal{Q} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$$

then ensures that the quadratic equation in $\dot{y}(0)$ given by (82) has no real solution.

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