

DOUBLE SINGULARITY-INDUCED BIFURCATION POINTS AND SINGULAR HOPF BIFURCATIONS

R.E.BEARDMORE

ABSTRACT. We explore the singularity-induced bifurcation and singular Hopf bifurcation theorems and the degeneracies that arise when Newton's Laws are coupled to Kirchhoff's Laws. Such models are used in the electrical engineering literature to describe electrical power systems and they can take the form of either an index-1 differential-algebraic equation (DAE) or a singularly perturbed ordinary differential equation (ODE). As a consequence of the debate in the engineering literature as to which class of system is the 'true' representation of power systems, we include a discussion of the consequences of the power engineer's 'load-flow singularity' for both ODE and DAE.

1. INTRODUCTION

This paper discusses the nature of singular Hopf curves and singularity-induced bifurcations (SIB) for those singular systems which possess a second-order structure. This is motivated by electrical power systems for the following reasons. In [15, 19, 8, 16] one finds that there is often a second-order, near-Hamiltonian structure present in the swing equations of classical power system models which can be exploited, using a Melnikov method, to demonstrate the existence of chaotic motion. One can also see from [8, 11, 20] that there has been a debate in the power engineering literature as to whether a power system is most faithfully represented by the index-1 differential-algebraic equation (DAE) or by a singularly perturbed ODE. The purpose of this paper is to place SIB [4] and singular Hopf bifurcation (SHB) theorems [3, 21] in the context of those singular systems which have a second-order, although not necessarily near-Hamiltonian structure.

There are essentially three objectives of this paper. Firstly we give a definition of singularity-induced bifurcation point and present a discussion of their existence in the class of DAE

$$(1) \quad \ddot{x} = f(x, \dot{x}, y, \lambda) \in \mathbb{R}^n, \quad 0 = g(x, y, \lambda) \in \mathbb{R}^m, \quad y \in \mathbb{R}^m$$

which arises in the theory of power systems [16, 17, 8]. One aspect of this system which makes it degenerate is the independence of the algebraic constraint of any velocity terms \dot{x} . The notion of SIB point, which first arose in [22], is usually discussed in terms of the DAE

$$(2) \quad \dot{x} = f(x, y, \lambda), \quad 0 = g(x, y, \lambda).$$

However, when we write (1) as a first-order system, it cannot satisfy the SIB theorem from [4].

Secondly, we extend the applicability of the singular Hopf bifurcation theorem found in [3, 21] for the system of ODEs

$$(3) \quad x_\tau = \epsilon f(x, y, \lambda, \epsilon) \in \mathbb{R}^n, \quad y_\tau = g(x, y, \lambda, \epsilon) \in \mathbb{R}^m,$$

independently of the slow-manifold dimension, n .

Here, f and g are smooth functions of all variables. We shall denote $x_\tau = \frac{dx}{d\tau}$ and $\dot{x} = \frac{dx}{dt}$ with $t = \epsilon\tau$ and a prime \cdot' will denote the partial derivative with respect to λ . We shall use $\mathbf{N}(L)$ to denote the null-space of the linear mapping L and $\mathcal{N}(L)$ is the generalised null-space of L . For a vector $v \in \mathbb{R}^p$ we write $\langle v \rangle = \mathbb{R} \cdot v = \{\mu v : \mu \in \mathbb{R}\}$. Also, $\mathcal{L}(V)$ denotes the space of all linear maps over some vector space V .

Thirdly, we adapt the proof of the SHB theorem to include the second-order structure observed in some power system models which have a *fast* rather than algebraic variable y

$$(4) \quad \ddot{x} = f(x, \dot{x}, y, \lambda, \epsilon) \in \mathbb{R}^n, \quad \epsilon \dot{y} = g(x, y, \lambda, \epsilon) \in \mathbb{R}^m.$$

We show how the existence of SIB points when $\epsilon = 0$ may signal the existence of Hopf bifurcation points in this system when $\epsilon \neq 0$.

Moreover, due to the local nature of the assumptions used, it is sufficient only to study the linearisations of (1),(3) and (4) about some equilibrium locus. The dynamical consequences are then immediate using standard, local bifurcation theory, subject to typical non-resonance conditions [2, 1].

In the last section of this paper we see the effects that SIB points can have in a power system model of the form (1) taken from the electrical engineering literature.

1.1. Preliminaries and Definitions.

1.1.1. *Hopf Curves.* Assume (3) has a trivial equilibrium for all $(\lambda, \epsilon) \in \mathbb{R}^2$. From [3, 21] we know, when $n = 1$, that the assumption

$$(5) \quad \mathbf{N}(d_y g(0, 0, \lambda_0, 0)) = \langle k \rangle \quad (k \in \mathbb{R}^m)$$

implies the existence of a smooth curve of so-called Hopf points $\{(\lambda(s), \epsilon(s)) \subset \mathbb{R}^2 : s \in I \subset \mathbb{R}\}$ in parameter space. This assumption, in terms of power engineering, is called the ‘load-flow singularity’ and it dominates the work of many industrial power engineers, as can be seen from the notes [14] which accompany a colloquium on this matter. The condition (5) is often cited as a possible cause of ‘voltage collapse’ whereby there is a catastrophic failure in a part of the electrical network. In DAE terms it is called a ‘solution-manifold singularity’ whereas in singularly-perturbed parlance it is described using the term ‘slow-manifold singularity’.

Suppose we define

$$K(\lambda, \epsilon) = \begin{pmatrix} \epsilon A(\lambda, \epsilon) & \epsilon B(\lambda, \epsilon) \\ C(\lambda, \epsilon) & D(\lambda, \epsilon) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \epsilon d_x f(0, 0, \lambda, \epsilon) & \epsilon d_y f(0, 0, \lambda, \epsilon) \\ d_x g(0, 0, \lambda, \epsilon) & d_y g(0, 0, \lambda, \epsilon) \end{pmatrix}.$$

We would like to ascertain the conditions under which there exists a smooth curve $s \mapsto \omega(s)$, for $s \in I \subset \mathbb{R}$, such that

$$\pm i\omega(s) \in \sigma(K(\lambda(s), \epsilon(s))).$$

Definition 1. *A smooth curve $\{(\lambda(s), \epsilon(s)) \in \mathbb{R}^2 : s \in I \subset \mathbb{R}\}$ is called a singular Hopf curve if $\epsilon(0) = 0$, $\epsilon(s) \neq 0$ for $s \neq 0$ and there is a smooth function $s \mapsto \omega(s) > 0$ such that for all $s \in I$*

$$i\omega(s) \in \sigma(K(\lambda(s), \epsilon(s))).$$

The existence of singular Hopf curves is, of course, related to the existence of periodic solutions and ducks oscillations [2, 21, 23, 3] but does not directly imply their existence without further transversality assumptions.

1.1.2. *Singularity-Induced Bifurcation.* Singularity-induced bifurcation is the term now used to describe when the linearisation of a DAE, with a parameter, has parameterised eigenvalues with poles with respect to that parameter. Typically for (2), if $\alpha(\lambda)$ is an eigenvalue then there is some $\mu \neq 0$ such that $\alpha(\lambda) \sim \mu(\lambda - \lambda_0)^{-1}$ as $\lambda \rightarrow \lambda_0$ at an SIB point [4, 22]. However, no definition of SIB point has appeared in the literature to date. So, to place it in a mathematical framework we give the following definition.

Definition 2 (SIB point). *Suppose that (2) has a trivial equilibrium for all $\lambda \in \mathbb{R}$ and that the linearisation about this equilibrium has an eigenvalue locus $\alpha(\lambda)$. If, for some sequences $(l_n), (\lambda_n) \subset \mathbb{R}$ with $l_n \uparrow \lambda_0$, $\lambda_n \downarrow \lambda_0$ one observes*

- (1) $\alpha(l_n)$ and $\alpha(\lambda_n) \rightarrow \infty$ as $n \rightarrow \infty$ such that
- (2) $\Re(\alpha(l_n))\Re(\alpha(\lambda_n)) < 0$

for all n , then λ_0 is said to be a singularity-induced bifurcation point.

If there are exactly p distinct eigenvalue loci with property 1 such that precisely one of them has property 2 then λ_0 is said to be a simple, double, triple, etc. SIB point according to the value of p .

One can see that if the DAE (2) satisfies the SIB theorem of [4, 22] at λ_0 then it has a simple SIB point at λ_0 .

Definition 2 is adopted for the following reasons. Property 1 is the essential diverging eigenvalue property of the original SIB theorem from [22], but the following DAE has a diverging eigenvalue and therefore satisfies property 1. However, it does not satisfy property 2.

Example 1. *Consider*

$$\dot{x} = -x - y, 0 = x - \lambda^2 y.$$

The eigenvalue is $-(1 + 1/\lambda^2)$.

There is no change in stability of the zero equilibrium of this system as λ passes through 0, so we choose to rule out this degenerate behaviour in the definition of SIB point. The following system shows why simple poles in eigenvalue loci are not sufficient to provide SIB points.

Example 2. *Consider*

$$\dot{x} = z, \dot{y} = x, 0 = y - \lambda^2 z.$$

There are two eigenvalue loci with simple poles, namely $\pm 1/\lambda$.

For all λ near 0, the dimension of the stable and unstable invariant subspaces $E^{s,u}(0)$ is one. Also, $\lambda = 0$ is not an SIB point according to Definition 2.

The capacity for DAEs to exhibit SIB points stems from the fact that matrix pencils admit infinite eigenvalues [13, 6] and parameterised matrix pencils are the linearisations of DAE. Notice from Definition 2 that no mention is made of any bounded eigenvalues and their behaviour at λ_0 ; we simply choose not to include this in our definition.

1.1.3. *Matrix Pencils and The Kronecker Normal Form.* Suppose that $(\hat{A}, \hat{B}) \in \mathcal{L}(\mathbb{R}^N) \times \mathcal{L}(\mathbb{R}^N)$ is a square matrix pair. It induces an affine mapping $\mathbb{R} \rightarrow \mathcal{L}(\mathbb{R}^N)$ by

$$s \mapsto s\hat{A} - \hat{B}.$$

Both the pair (\hat{A}, \hat{B}) and the induced mapping are said to be *matrix pencils*. The pencil is said to be *regular* if and only if there exists an $s_0 \in \mathbb{C}$ such that $\det(s_0\hat{A} - \hat{B}) \neq 0$.

We say that the spectrum of the matrix pencil is given by

$$\sigma(\hat{A}, \hat{B}) = \{s \in \mathbb{C} : \det(s\hat{A} - \hat{B}) = 0\}.$$

Denote by $\#\sigma(\hat{A}, \hat{B})$ the cardinality of the spectrum and note that this could be zero.

The following theorem [13, 6] is the so-called Kronecker Normal Form. This provides the matrix pencil analogy of the Jordan Normal Form.

Theorem 1 (Kronecker Normal Form (KNF)). *Suppose that (\hat{A}, \hat{B}) is a regular matrix pencil on \mathbb{R}^N . One can decompose $\mathbb{R}^N = U \oplus V$ where there are maps $P \in GL(\mathbb{R}^N)$, $Q \in GL(U \oplus V, \mathbb{R}^N)$ and a $\hat{C} \in \mathcal{L}(U)$ such that*

$$P\hat{A}Q = \begin{bmatrix} I_u & 0 \\ 0 & N \end{bmatrix}, \quad P\hat{B}Q = \begin{bmatrix} \hat{C} & 0 \\ 0 & I_v \end{bmatrix}$$

where there is a $\nu \geq 1$ such that $N^\nu = 0$. Let ν be the smallest integer such that $N^\nu = 0$. Here $I_u \in \mathcal{L}(U)$ and $N, I_v \in \mathcal{L}(V)$; I_u and I_v denote identities. Moreover,

$$\sigma(\hat{A}, \hat{B}) = \sigma(\hat{C}) \quad \text{and} \quad \#\sigma(\hat{A}, \hat{B}) = \dim U.$$

The pair (\hat{A}, \hat{B}) has index given by $\text{ind}(\hat{A}, \hat{B}) = \nu$.

2. DOUBLE SIB POINTS

Let us examine the singular behaviour along an equilibrium locus in the following second-order, index-1 DAE

$$(6) \quad \dot{x} = f(x, y, \lambda), \quad 0 = g(x, y, \lambda).$$

This is reversible with involution $(x, \dot{x}, y) \mapsto (x, -\dot{x}, y)$. Assume (6) has a trivial equilibrium for all $\lambda \in \mathbb{R}$ and write its linearisation as the matrix pencil $(M, L(\lambda))$, where

$$M \stackrel{\text{def}}{=} \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{2n+m})$$

and

$$L(\lambda) \stackrel{\text{def}}{=} \left(\begin{array}{cc|c} 0 & I & 0 \\ \mathcal{A}(\lambda) & 0 & \mathcal{B}(\lambda) \\ \hline \mathcal{C}(\lambda) & 0 & \mathcal{D}(\lambda) \end{array} \right) \stackrel{\text{def}}{=} \left(\begin{array}{cc|c} 0 & I & 0 \\ f_x & 0 & f_y \\ \hline g_x & 0 & g_y \end{array} \right) \in \mathcal{L}(\mathbb{R}^{2n+m}),$$

evaluated at $(0, 0, \lambda)$.

Lemma 1 (Reversible SIB). *Suppose $\mathbf{N}(D(\lambda_0)) = \langle k \rangle$ for some non-zero $k \in \mathbb{R}^m$ and that $\mathbf{N}(D(\lambda_0)^T) = \langle u \rangle$. Let $\delta_0 \stackrel{\text{def}}{=}} u^T \mathcal{D}'(\lambda_0)k \neq 0$ and $\mu_0 \stackrel{\text{def}}{=}}$ $u^T \mathcal{C}(\lambda_0)\mathcal{B}(\lambda_0)k \neq 0$. It follows that the matrix pencil $(M, L(\lambda))$ has two algebraically simple eigenvalues, $\alpha_{\pm}(\lambda)$, such that*

$$\alpha_{\pm}(\lambda) = \pm \frac{\sqrt{\delta_0/\mu_0}}{\sqrt{\lambda - \lambda_0}} \pm \psi(\sqrt{\lambda - \lambda_0})$$

for all λ near λ_0 , where $\psi(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$ and $\psi(x) = O(x)$ as $x \rightarrow 0$.

Proof. By definition, $\sigma(M, L(\lambda)) = \{\alpha \in \mathbb{C} : \exists v \neq 0, \alpha Mv = L(\lambda)v\}$. It follows that, if $\mathcal{D}(\lambda) = g_y(0, 0, \lambda)$ is invertible, the eigenvalues of the linearisation about the equilibrium of (6) are the eigenvalues of the Schur complement

$$\mathbf{S}_{\lambda} = \begin{pmatrix} 0 & I \\ (\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})(\lambda) & 0 \end{pmatrix}.$$

So define $\mathcal{S}(\lambda) \stackrel{\text{def}}{=}}$ $(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})(\lambda)$. It is clear from

$$\mathbf{S}_{\lambda}^2 = \begin{pmatrix} \mathcal{S}(\lambda) & 0 \\ 0 & \mathcal{S}(\lambda) \end{pmatrix}$$

that $\sigma(M, L(\lambda)) = \sigma(\mathbf{S}_{\lambda}) = +\sqrt{\sigma(\mathcal{S}(\lambda))} \cup -\sqrt{\sigma(\mathcal{S}(\lambda))}$. One can now apply SIB from [4, 5] to the Schur complement $\mathcal{S}(\lambda)$ to finish the proof. \square

This lemma shows that the reversible structure in (6) forces poles of order 1/2 in the eigenvalues of its linearisation. This is not merely of academic interest since this class of model has been used to describe power systems [16]. However, λ_0 is not an SIB point according to Definition 2. This is

because property 2 is not satisfied and one cannot decide on the basis of the eigenvalues whether or not their divergence has led to a change in the local stability properties of the equilibrium locus.

More generally, one can generate poles of order $1/j$ in eigenvalue loci, for any $j \in \mathbb{N}$, by applying the SIB theorem to the following DAE

$$(7) \quad \left(\frac{d}{dt} \right)^j x = \mathcal{A}(\lambda)x + \mathcal{B}(\lambda)y, \quad 0 = \mathcal{C}(\lambda)x + \mathcal{D}(\lambda)y.$$

If the matrix pencil

$$(\mathcal{M}, \mathcal{L}(\lambda)) \stackrel{\text{def}}{=} \left(\left(\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{array} \right) \right) \in \mathcal{L}(\mathbb{R}^{n+m}) \times \mathcal{L}(\mathbb{R}^{n+m})$$

satisfies the SIB theorem of [4] at $\lambda = \lambda_0$, then there are diverging eigenvalues of the j -th order problem (7) given by the j complex numbers $\left(\frac{\mu}{\lambda - \lambda_0} + O(1) \right)^{1/j}$. One can subsequently show that there are DAEs with SIB points of order j , for any natural number j .

2.1. Including Dissipative Terms. We now consider the nature of SIB points in the following class of index-1 DAE,

$$(8) \quad \ddot{x} = \mathcal{A}(\lambda)x + \Theta(\lambda)\dot{x} + \mathcal{B}(\lambda)y, \quad 0 = \mathcal{C}(\lambda)x + \mathcal{D}(\lambda)y.$$

Define

$$L_d(\lambda) \stackrel{\text{def}}{=} \left(\begin{array}{cc|c} 0 & I & 0 \\ \mathcal{A}(\lambda) & \Theta(\lambda) & \mathcal{B}(\lambda) \\ \hline \mathcal{C}(\lambda) & 0 & \mathcal{D}(\lambda) \end{array} \right) \stackrel{\text{def}}{=} \left(\begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right) \in \mathcal{L}(\mathbb{R}^{2n+m}).$$

Clearly if $D (= \mathcal{D})$ is invertible then

$$\sigma(M, L_d(\lambda)) = \sigma(A - BD^{-1}C) = \sigma \left(\begin{array}{cc} 0 & I \\ \mathcal{A}(\lambda) - \mathcal{B}(\lambda)\mathcal{D}(\lambda)^{-1}\mathcal{C}(\lambda) & \Theta(\lambda) \end{array} \right).$$

To analyse the spectrum of this pencil at infinity it is simpler to consider a related problem at zero, assuming that $L_d(\lambda_0)$ is invertible, as the following lemma from [5] shows.

Lemma 2. *Define the matrices*

$$M = \left(\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right) \text{ and } L = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in \mathcal{L}(\mathbb{R}^{p+q})$$

for $p, q \in \mathbb{N}$ and suppose that $\det L \cdot \det D \neq 0$. If

$$L^{-1} = \left(\begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right) \in \mathcal{L}(\mathbb{R}^{p+q})$$

and

$$\mathbf{S}_\lambda = A - BD^{-1}C \in \mathcal{L}(\mathbb{R}^p)$$

then $\mathbf{S}_\lambda = A_1^{-1}$ and $\det L = \det \mathbf{S}_\lambda \cdot \det D$. Moreover, if $\det D = 0$ then $\det A_1 = 0$. If $\mathbf{N}(D) = \langle k \rangle$ and $CBk \notin \mathbf{R}(D)$ then $\mathbf{N}(A_1) = \langle Bk \rangle$ and $Bk \notin \mathbf{R}(A_1)$. In addition, $\sigma(M, L) = 1/(\sigma(A_1) \setminus \{0\})$.

Denote

$$L_d(\lambda)^{-1} = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{pmatrix} \in GL(\mathbb{R}^{2n+m})$$

and write

$$\begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{A}_1(\lambda) & \mathcal{B}_1(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{D}_1(\lambda) \end{pmatrix} \in GL(\mathbb{R}^{n+m}).$$

Using Lemma 2, if \mathcal{D} ($= D$) is invertible then \mathcal{A}_1 is invertible,

$$A_1 = \mathbf{S}_\lambda^{-1} = (A - BD^{-1}C)^{-1} \text{ and } S_\lambda \stackrel{\text{def}}{=} A_1^{-1} = (A - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1}$$

and therefore for λ near λ_0 ,

$$\sigma(\mathbf{S}_\lambda) = \sigma(M, L_d(\lambda)) = 1/(\sigma(A_1(\lambda)) \setminus \{0\}).$$

From these observations one can deduce the following lemma. This gives information as to both the divergent and bounded elements of $\sigma(M, L_d(\lambda))$ at a point where D ceases to be invertible.

Lemma 3. *Suppose that $\det L_d(\lambda_0) \neq 0$, $\mathbf{N}(D(\lambda_0)) = \langle k \rangle$ for some non-zero $k \in \mathbb{R}^m$, $D'(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$ and $\mathcal{C}(\lambda_0)\mathcal{B}(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$. It follows that $A_1(\lambda_0)$ is a singular mapping with a zero eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1. Hence, there are $2n - 2$ eigenvalues of $(A - BD^{-1}C)(\lambda)$ which can be extended to non-zero, continuous functions of λ in a neighbourhood of λ_0 .*

Proof. A simple calculation shows that if

$$\begin{pmatrix} \mathcal{A}(\lambda_0) & \mathcal{B}(\lambda_0) \\ \mathcal{C}(\lambda_0) & \mathcal{D}(\lambda_0) \end{pmatrix}^{-1} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{C}_1 & \mathcal{D}_1 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{n+m})$$

then

$$L_d(\lambda_0)^{-1} = \left(\begin{array}{cc|c} -\mathcal{A}_1\Theta & \mathcal{A}_1 & \mathcal{B}_1 \\ I_n & 0 & 0 \\ \hline -\mathcal{C}_1\Theta & \mathcal{C}_1 & \mathcal{D}_1 \end{array} \right) = \left(\begin{array}{c|c} \mathcal{A}_1 & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \end{array} \right) \in \mathcal{L}(\mathbb{R}^{2n+m})$$

so that

$$A_1 = A_1(\lambda_0) = \lim_{\lambda \rightarrow \lambda_0} (A - BD^{-1}C)^{-1} = \left(\begin{array}{cc} -\mathcal{A}_1\Theta & \mathcal{A}_1 \\ I_n & 0 \end{array} \right) \Big|_{\lambda=\lambda_0},$$

where $\Theta(\lambda_0) = \Theta$. Under the assumptions in this lemma it follows that

$$\mathbf{N}(A_1) = \langle \mathcal{B}k \rangle$$

which represents an algebraically simple, zero eigenvalue of A_1 . Note also that $\lim_{\lambda \rightarrow \lambda_0} (A - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1} = A_1(\lambda_0)$.

Suppose that for some $(u, v) \in \mathbb{R}^{n+m}$

$$A_1^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using $\mathbf{N}(A_1) = \langle (0, \mathcal{B}k) \rangle$ a direct calculation yields $v - \Theta u = r\mathcal{B}k$, for any $r \in \mathbb{R}$. Therefore $u = s\mathcal{B}k$ and $v = r\mathcal{B}k + s\Theta\mathcal{B}k$ for any $s \in \mathbb{R}$. Hence (u, v) lies in the two-dimensional space

$$\mathcal{V} \stackrel{\text{def}}{=} \langle (0, \mathcal{B}k), (\mathcal{B}k, \Theta\mathcal{B}k) \rangle.$$

One can easily show that $\mathbf{N}(A_1^3) = \mathbf{N}(A_1^2) = \mathcal{V}$, so that $\dim \mathcal{N}(A_1) = 2$ and the first part of the lemma is proven.

To prove the remainder, simply note that because A_1 has a zero eigenvalue of algebraic multiplicity equal to 2, the remaining $2n - 2$ eigenvalues in $\sigma(A_1)$ are non-zero. As $A_1 = (A - BD^{-1}C)^{-1}$, the reciprocals of these $2n - 2$ non-zero eigenvalues provide the necessary numbers to remove the singularities from the bounded elements of the spectrum of $A - BD^{-1}C$ at $\lambda = \lambda_0$. \square

This implies the existence of $2n$ functions, $\alpha_j(\lambda) \in \sigma(M, L_d(\lambda))$, such that as $\lambda \rightarrow \lambda_0$

$$\alpha_{1,2}(\lambda) \rightarrow \infty, \alpha_j(\lambda) \rightarrow \alpha_j \neq 0 \quad (j = 3, \dots, 2n).$$

We find that λ_0 satisfies property 1 of Definition 2. To prove λ_0 is a double SIB point we need the real part of the two diverging eigenvalues $\alpha_{1,2}$ and this we now calculate.

Lemma 4. *If we denote, under the assumptions of Lemma 3*

$$\mu = -\frac{u^T \mathcal{C}(\lambda_0) \mathcal{B}(\lambda_0) k}{u^T \mathcal{D}'(\lambda_0) k} \quad (u \in \mathbf{N}(\mathcal{D}(\lambda_0)^T))$$

then $\lim_{\lambda \rightarrow \lambda_0} \alpha_{1,2}(\lambda)^2 (\lambda - \lambda_0) = \mu$.

Proof. One can apply the argument from the proof of the SIB theorem in [4, 5] to the mapping $J(\lambda) \stackrel{\text{def}}{=} (\lambda - \lambda_0) \mathbf{S}_\lambda^2$. Note that for all λ near λ_0

$$(\lambda - \lambda_0) \alpha_{1,2}(\lambda)^2 \in \sigma(J(\lambda)).$$

Define

$$J(\lambda_0) = \begin{pmatrix} \mathcal{J}_0 & 0 \\ \Theta \mathcal{J}_0 & \mathcal{J}_0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{2n})$$

where $\mathcal{J}_0 = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)(A - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}) = -\frac{1}{d} \mathcal{B}(\text{adj} \mathcal{D}) \mathcal{C}$ and $d = \frac{d}{d\lambda} \det \mathcal{D}(\lambda) \big|_{\lambda=\lambda_0}$. From the proof of the SIB theorem in [5] we see that $\sigma(\mathcal{J}_0) = \{0, \mu\}$ where $\mathcal{J}_0 \mathcal{B}k = \mu \mathcal{B}k$ and the result follows. \square

We now find sufficient conditions for the existence of double SIB points in (8).

Lemma 5. *Suppose that the assumptions of Lemma 3 apply to $(M, L_d(\lambda))$ at $\lambda = \lambda_0$ and let $\alpha_{1,2}(\lambda)$ be the two diverging eigenvalues of \mathbf{S}_λ at λ_0 . Since $\text{tr}(\mathbf{S}_\lambda) = \text{tr}(\Theta(\lambda))$ is a C^1 function in a neighbourhood of λ_0 , if we denote $\cup_{i=1}^{2n} \alpha_i(\lambda) = \sigma(\mathbf{S}_\lambda)$, we may define the real-valued continuous functions*

$$\tau(\lambda) \stackrel{\text{def}}{=} -\frac{\det L_d(\lambda_0)}{d} \prod_{i=3}^{2n} \alpha_i(\lambda)^{-1},$$

with $d = \frac{d}{d\lambda} \det \mathcal{D}(\lambda) \Big|_{\lambda=\lambda_0} \neq 0$ and

$$q(\lambda) \stackrel{\text{def}}{=} \text{tr}(\Theta(\lambda)) - \sum_{i=3}^{2n} \alpha_i(\lambda).$$

Then $\tau(\lambda_0) = -\frac{\det L_d(\lambda_0)}{d \prod_{j=3}^{2n} \alpha_j} \neq 0$ and there admits an asymptotic representation of two algebraically simple eigenvalues $\alpha_{1,2}(\lambda)$ of \mathbf{S}_λ given by

$$\alpha_{1,2}(\lambda) \sim \pm \frac{\sqrt{\tau(\lambda)}}{\sqrt{\lambda - \lambda_0}} + \frac{1}{2}q(\lambda_0) + O\left(\sqrt{\lambda - \lambda_0}\right)$$

as $\lambda \rightarrow \lambda_0$.

Proof. The $2n - 2$ continuous functions $\alpha_i(\lambda)$, $i = 3, \dots, 2n$, have removable singularities at λ_0 with values which are non-zero. By Lemma 2, $\prod_{i=1}^{2n} \alpha_i(\lambda) = \frac{\det L(\lambda)}{\det D(\lambda)}$. Therefore, for all λ near λ_0 with $\lambda \neq \lambda_0$, we have

$$\alpha_1(\lambda)\alpha_2(\lambda) = \frac{\det L(\lambda)}{\det D(\lambda) \prod_{i=3}^{2n} \alpha_i(\lambda)} \stackrel{\text{def}}{=} p(\lambda) \text{ and } \alpha_1(\lambda) + \alpha_2(\lambda) = q(\lambda).$$

It follows easily that $\alpha(\lambda)^2 - \alpha(\lambda)q(\lambda) + p(\lambda) = 0$ from where $\alpha_{1,2}(\lambda) = \frac{1}{2} \left\{ q(\lambda) \pm [q(\lambda)^2 - 4p(\lambda)]^{1/2} \right\}$. But $\det D(\lambda) = d \cdot (\lambda - \lambda_0) + o(\lambda - \lambda_0)$ and $p(\lambda) = \frac{-\tau(\lambda)}{\lambda - \lambda_0} + O(1)$ as $\lambda \rightarrow \lambda_0$. The asymptotic representation of the eigenvalues now follows. Algebraic simplicity of $\alpha_{1,2}$ follows from the existence of α_j for $j = 3, \dots, 2n$ and the subsequent boundedness of $\alpha_j(\lambda)$ at λ_0 , for $j = 3, \dots, 2n$. \square

Definition 3. For a square matrix pencil (\hat{A}, \hat{B}) , define its trace

$$\text{tr}(\hat{A}, \hat{B}) = \sum \sigma(\hat{A}, \hat{B}).$$

The sum is taken over the elements of the finite spectrum and $\Sigma(\emptyset) = 0$.

We now state the main result of this section.

Theorem 2 (Double-SIB Theorem). *Suppose that the conditions of Lemma 3 apply to the DAE (8) at $\lambda = \lambda_0$ and*

$$(9) \quad \text{tr}(\Theta(\lambda_0)) \neq \text{tr}(M, L_d(\lambda_0)),$$

then λ_0 is a double SIB point.

Proof. Let $\lambda_n \uparrow \lambda_0$ and $\lambda_n \downarrow \lambda_0$ be any two sequences and write $q_0 = q(\lambda_0) \neq 0$ from Lemma 5. We know that $\alpha_{1,2}$ are the two diverging eigenvalues associated with (8) and any other eigenvalue remains bounded and non-zero at λ_0 . In the notation of Lemmas 3 and 5 suppose that $\tau(\lambda_0) > 0$; the $\tau(\lambda_0) < 0$ case is treated analogously.

As $n \rightarrow \infty$

$$\Re(\alpha_1(l_n)) \sim \Re\left(\frac{\sqrt{\tau(l_n)}}{\sqrt{l_n - \lambda_0}} + \frac{1}{2}q_0 + O(l_n - \lambda_0)^{1/2}\right) \rightarrow \frac{1}{2}q_0,$$

$$\Re(\alpha_1(\lambda_n)) = \frac{\sqrt{\tau(\lambda_n)}}{\sqrt{\lambda_n - \lambda_0}} \Re\left(1 + O(\lambda_n - \lambda_0)^{1/2}\right),$$

$$\Re(\alpha_2(l_n)) \sim \Re\left(-\frac{\sqrt{\tau(l_n)}}{\sqrt{l_n - \lambda_0}} + \frac{1}{2}q_0 + O(l_n - \lambda_0)^{1/2}\right) \rightarrow \frac{1}{2}q_0,$$

and

$$\Re(\alpha_2(\lambda_n)) = \frac{\sqrt{\tau(\lambda_n)}}{\sqrt{\lambda_n - \lambda_0}} \Re\left(-1 + O(\lambda_n - \lambda_0)^{1/2}\right).$$

Now

$$\Re(\alpha_1(l_n))\Re(\alpha_1(\lambda_n)) \sim \frac{1}{2}q_0 \left(\frac{\tau(\lambda_0)}{\lambda_n - \lambda_0}\right)^{1/2} + O(1),$$

and

$$\Re(\alpha_2(l_n))\Re(\alpha_2(\lambda_n)) \sim -\frac{1}{2}q_0 \left(\frac{\tau(\lambda_0)}{\lambda_n - \lambda_0}\right)^{1/2} + O(1);$$

exactly one of these is negative for all n large enough and the result follows. \square

This theorem provides exactly the right degeneracy condition to ensure that (1) has a double SIB point in the sense of Definition 2. If condition (9) does not hold there is no reason to expect property 2 in Definition 2 to be satisfied. For instance, we find that the DAE (6) cannot have a double SIB point because the term $q(\lambda_0) = \text{tr}(\Theta(\lambda_0)) - \text{tr}(M, L_d(\lambda_0))$ is found to be zero for this class of DAE, as the following lemma shows.

Lemma 6. *Given the non-singular matrix*

$$L = \begin{bmatrix} 0 & I & 0 \\ \mathcal{A} & 0 & \mathcal{B} \\ \mathcal{C} & 0 & \mathcal{D} \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{2n+m})$$

then $\text{tr}(M, L) = 0$. If $\det \mathcal{D} = 0$ then $\#\sigma(M, L) \leq 2n - 2$. If $\mathbf{N}(\mathcal{D}) = \langle k \rangle$ and $\mathcal{C}\mathcal{B}k \notin \mathbf{R}(\mathcal{D})$ then $\#\sigma(M, L) = 2n - 2$.

Proof. Observe that $\sigma(M, L) = \{\sigma(L^{-1}M) \setminus \{0\}\}^{-1}$ and $L^{-1}M$ has the form

$$L^{-1}M = \begin{bmatrix} 0 & \mathcal{A}_1 & 0 \\ I & 0 & 0 \\ 0 & * & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{2n+m})$$

where \mathcal{A}_1 is defined in Lemma 3. The non-zero eigenvalues of this mapping are given by the non-zero elements of $\pm\sqrt{\sigma(\mathcal{A}_1)}$ and therefore $\alpha \in \sigma(M, L)$ implies $-\alpha \in \sigma(M, L)$ and the first part follows. If $\det \mathcal{D} = 0$, using Lemma 2, we know that $\det \mathcal{A}_1 = 0$ and the second part follows. Finally, under the conditions of Lemma 3 equality is achieved in the given inequality with

$\Theta(\lambda_0) = 0$ in the notation of Lemma 3. \square

2.2. Kronecker Index Jumps. An observation originally made in [5] is that the SIB theorem implies the existence of a jump in index of the pencil under consideration at the SIB point. Now, the structure associated with (1) is particularly special and the transition through the SIB point as λ varies is always from the class of index-1 matrix pencils, and back. The passage at the SIB point is via a class of higher (≥ 2) index pencils as shown by the two following lemmas.

Lemma 7 (Kronecker Index Jump [5]). *Denote*

$$M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{p+q})$$

and suppose $\det L \neq 0$. Then

$$\text{ind}(M, L) = 1 \Leftrightarrow \det D \neq 0.$$

If $\mathbf{N}(D) = \langle k \rangle$ for some non-zero $k \in \mathbb{R}^q$ such that $CBk \notin \mathbf{R}(D)$ then $\text{ind}(M, L) = 2$.

Lemma 8. *Suppose that (\hat{A}, \hat{B}) is a regular matrix pencil of index ν and ϕ is C^∞ , then the index of the non-autonomous, j -th order descriptor system*

$$(10) \quad \hat{A} \left(\frac{d}{dt} \right)^j z(t) = \hat{B}z(t) + \phi(t)$$

is $j(\nu - 1) + 1$.

Proof. Let Φ be a smooth function and (S, T) a regular matrix pencil. Suppose the solution operator of the DAE

$$(11) \quad S\dot{w} = Tw + \Phi(t)$$

is a function of the κ -jet $(t, \Phi, \Phi', \dots, \Phi^{(\kappa)})$ but *not* of the $\kappa+1$ -jet $(t, \Phi, \Phi', \dots, \Phi^{(\kappa+1)})$, then (11) has differential index $\kappa + 1$. The pencil (S, T) therefore has Kronecker index $\kappa + 1$. Now the solution of (10) can be written down in terms of ϕ using the Kronecker Normal Form. From [6] we find that the highest order derivative of ϕ which can appear is the $j(\nu - 1)$ -th and the result follows. \square

We can now see that the DAE (7) has a jump in its index at the SIB point λ_0 . If we write (7) in the form

$$\left(\frac{d}{dt} \right)^j \mathcal{M}\dot{z} = \mathcal{L}(\lambda)z$$

then we may apply Lemmas 7 and 8 as follows. At $\lambda = \lambda_0$, the j -th order system (7) has index

$$j \cdot (\text{ind}(\mathcal{M}, \mathcal{L}(\lambda_0)) - 1) + 1 = j + 1$$

because $\text{ind}(\mathcal{M}, \mathcal{L}(\lambda_0)) = 2$. If $\lambda \neq \lambda_0$ then $\text{ind}(\mathcal{M}, \mathcal{L}(\lambda)) = 1$, which implies that the index of (7) is also 1 when $\lambda \neq \lambda_0$.

3. SINGULAR PERTURBATIONS, SIMPLE & DOUBLE SIB POINTS

We can give an interpretation of double SIB points in terms of Hopf bifurcations for the related singular perturbation problem (4). Before doing so, we present a proof of the SHB theorem. This can then be easily generalised to include the structure present in (4).

3.1. The SHB Theorem. Now we give a simple proof of the existence of Hopf curves in (3) independently of n and m . Recall that the matrix $K(\lambda, \epsilon)$ is the linearisation of (3) about its trivial equilibrium locus.

Theorem 3. *Suppose that $\mathbf{N}(D(\lambda_0, 0)) = \langle k \rangle$ for some non-zero $k \in \mathbb{R}^m$ and $\sigma(D(\lambda_0, 0))$ contains no other eigenvalues of zero real part. Denote $\mathbf{N}(D(\lambda_0, 0)^T) = \langle u \rangle$ and suppose $\delta_0 \stackrel{\text{def}}{=} u^T D'(\lambda_0, 0)k \neq 0$. If¹ $u^T k = 1$ and*

$$u^T C(\lambda_0, 0)B(\lambda_0, 0)k = -\omega_0^2 < 0$$

then there is a $\delta > 0$ and a smooth curve $w \mapsto (\lambda_0(w), \epsilon_0(w))$, for $w \in (-\delta, \delta)$, such that

$$\pm i \frac{\epsilon_0(w)}{w} \in \sigma(K(\lambda_0(w), \epsilon_0(w))).$$

Moreover, this represents an algebraically simple eigenvalue. In addition,

$$\epsilon_0(0) = 0, \lambda_0(0) = \lambda_0, \lim_{w \rightarrow 0^+} \frac{\epsilon_0(w)}{w^2} = \omega_0^2$$

and both λ_0 and ϵ_0 are even functions.

Proof. Consider the suspended nonlinear system

$$(12) \quad \begin{aligned} \dot{x} &= \epsilon A(\lambda, \epsilon)x + \epsilon B(\lambda, \epsilon)y, \quad \dot{y} = C(\lambda, \epsilon)x + D(\lambda, \epsilon)y \\ \dot{\lambda} &= 0, \quad \dot{\epsilon} = 0. \end{aligned}$$

Define $D \stackrel{\text{def}}{=} D(\lambda_0, 0)$. By assumption we have $\mathbf{N}(D) = \langle k \rangle$ and $k \notin \mathbf{R}(D)$. If $\delta(\lambda, \epsilon) \stackrel{\text{def}}{=} u^T D(\lambda, \epsilon)k$ then $\delta_0 = \frac{\partial \delta}{\partial \lambda}(\lambda_0, 0) = u^T D'(\lambda_0, 0)k \neq 0$. Decompose $\mathbb{R}^m = \langle k \rangle \oplus \langle u \rangle^\perp$ and write

$$y(t) = \alpha(t)k + r(t), \quad r(t)^T u \equiv 0.$$

Then $\dot{y} = \dot{\alpha}k + \dot{r} = C(\lambda, \epsilon)x + D(\lambda, \epsilon)(\alpha k + r)$, and if $\Pi : \mathbb{R}^m \rightarrow \mathbf{R}(D)$ is the projection onto $\langle u \rangle^\perp = \mathbf{R}(D)$ along $\langle k \rangle$ then

$$\begin{aligned} \dot{\alpha} &= u^T C(\lambda, \epsilon)x + \delta(\lambda, \epsilon)\alpha + u^T D(\lambda, \epsilon)r \\ \dot{r} &= \Pi[C(\lambda, \epsilon)x + \alpha D(\lambda, \epsilon)k + D(\lambda, \epsilon)r]. \end{aligned}$$

¹This is a normalisation condition which does not affect the generality of the proof but implies that 0 is an algebraically simple eigenvalue of $D(\lambda_0, 0)$.

Write the restricted mapping $\Delta \stackrel{\text{def}}{=} D|_{\langle u \rangle^\perp} \in GL(\mathbf{R}(D))$, let $C \stackrel{\text{def}}{=} C(\lambda_0, 0)$ and make the linear transformation $\kappa = r + \Delta^{-1}\Pi Cx$. Then

$$\begin{aligned} \dot{\kappa} &= \dot{r} + \Delta^{-1}\Pi C\dot{x} \\ &= \Pi [C(\lambda, \epsilon)x + \alpha D(\lambda, \epsilon)k + D(\lambda, \epsilon)(\kappa - \Delta^{-1}\Pi Cx)] + \\ &\quad \epsilon \Delta^{-1}\Pi C [A(\lambda, \epsilon)x + B(\lambda, \epsilon)[\alpha k + \kappa - \Delta^{-1}\Pi Cx]] \end{aligned}$$

so that when $\epsilon = 0, \lambda = \lambda_0$ we have the linear system

$$\dot{\epsilon} = 0, \dot{\lambda} = 0, \dot{x} = 0, \dot{\alpha} = u^T Cx, \dot{\kappa} = \Delta \kappa.$$

By the centre manifold theorem, using the fact that Δ is hyperbolic, one can find a smooth invariant manifold on which $\kappa = h(x, \alpha, \lambda, \epsilon)$ and therefore $r = -\Delta^{-1}\Pi Cx + h(x, \alpha, \lambda, \epsilon)$. It is also true that $h(0, 0, \lambda_0, 0) = 0$ and $dh(0, 0, \lambda_0, 0) = 0$.

We now consider the reduced system on the centre manifold, which is

$$(13) \quad \begin{aligned} \dot{x} &= \epsilon \hat{A}(\lambda, \epsilon)x + \epsilon \hat{B}(\lambda, \epsilon)\alpha + h_1(x, \alpha, \lambda, \epsilon) \\ \dot{\alpha} &= \hat{C}(\lambda, \epsilon)x + \delta(\lambda, \epsilon)\alpha + h_2(x, \alpha, \lambda, \epsilon). \end{aligned}$$

Note that

$$\begin{aligned} \hat{A}(\lambda, \epsilon) &= A(\lambda, \epsilon) - B(\lambda, \epsilon)\Delta^{-1}\Pi C, \quad \hat{B}(\lambda, \epsilon) = B(\lambda, \epsilon)k \\ \hat{C}(\lambda, \epsilon) &= u^T [C(\lambda, \epsilon) - D(\lambda, \epsilon)\Delta^{-1}\Pi C], \quad h_1 = B(\lambda, \epsilon)h, \quad h_2 = u^T D(\lambda, \epsilon)h. \end{aligned}$$

Now seek conditions which ensure that equation (13) has a linearisation with purely imaginary eigenvalues. To this end, scale the linearisation of (13) about the equilibrium $x = 0, \alpha = 0$ by $1/\epsilon$ and seek a $\mu \in \mathbb{C}$ such that

$$0 = c(\mu, \lambda, \epsilon) \stackrel{\text{def}}{=} \left| \begin{array}{c|c} \hat{A}(\lambda, \epsilon) - \mu I_n & \hat{B}(\lambda, \epsilon) \\ \hline \epsilon^{-1}\hat{C}(\lambda, \epsilon) & \epsilon^{-1}\delta(\lambda, \epsilon) - \mu \end{array} \right|.$$

The eigenvalues associated with the linearisation of (13) are $\mu\epsilon$ when μ solves $c(\mu, \lambda, \epsilon) = 0$. Now, let $\mu = i/w$ for $w \in \mathbb{R}$ and seek a solution to $c(\mu, \lambda, \epsilon) = 0$ for $|w|$ small. Using Lemma 2,

$$\begin{aligned} c(\mu, \lambda, \epsilon) &= \det \left(\hat{A}(\lambda, \epsilon) - \mu I_n \right) \times \\ &\quad \epsilon^{-1} \left(\delta(\lambda, \epsilon) - \epsilon\mu - \hat{C}(\lambda, \epsilon)[\hat{A}(\lambda, \epsilon) - \mu I_n]^{-1}\hat{B}(\lambda, \epsilon) \right) \end{aligned}$$

so that, for small enough $|w|$, $c(i/w, \lambda, \epsilon) = 0$ if and only if

$$0 = -w\delta(\lambda, \epsilon) + \epsilon i + w^2 \hat{C}(\lambda, \epsilon)[w\hat{A}(\lambda, \epsilon) - iI_n]^{-1}\hat{B}(\lambda, \epsilon).$$

This is true because the smoothly-parameterised linear mapping $L(w, \lambda, \epsilon) = w\hat{A}(\lambda, \epsilon) - iI_n \in \mathcal{L}(\mathbb{C}^n)$ satisfies $L(0, \lambda, \epsilon) \in GL(\mathbb{C}^n)$ and therefore $L(w, \lambda, \epsilon) \in GL(\mathbb{C}^n)$ for all (w, λ, ϵ) in some neighbourhood of $(0, \lambda_0, 0)$. By the elementary Banach lemma

$$(14) \quad L(w, \lambda, \epsilon)^{-1} = i(I_n + iw\hat{A}(\lambda, \epsilon))^{-1} = i \sum_{j=0}^{\infty} (-i)^j w^j \hat{A}(\lambda, \epsilon)^j,$$

provided $|w| \cdot \|\hat{A}(\lambda, \epsilon)\| < 1$ in some norm. It follows that the real and imaginary parts of the equation $c(i/w, \lambda, \epsilon) = 0$ are given by

$$(150) = -\delta(\lambda, \epsilon) + w^2 \hat{C}(\lambda, \epsilon) \sum_{j=0}^{\infty} \hat{A}(\lambda, \epsilon)^{4j+1} w^{4j} [I_n - \hat{A}(\lambda, \epsilon)^2 w^2] \hat{B}(\lambda, \epsilon)$$

$$(160) = \epsilon + w^2 \hat{C}(\lambda, \epsilon) \sum_{j=0}^{\infty} \hat{A}(\lambda, \epsilon)^{4j} w^{4j} [I_n - \hat{A}(\lambda, \epsilon)^2 w^2] \hat{B}(\lambda, \epsilon)$$

This defines a system $\gamma(w, \lambda, \epsilon) = 0 \in \mathbb{R}^2$ such that $\gamma(0, \lambda_0, 0) = 0$ and the derivative of γ with respect to (λ, ϵ) is

$$d_{(\lambda, \epsilon)} \gamma(0, \lambda_0, 0) = \begin{pmatrix} -\frac{\partial \delta}{\partial \lambda}(\lambda_0, 0) & -\frac{\partial \delta}{\partial \epsilon}(\lambda_0, 0) \\ 0 & 1 \end{pmatrix}.$$

This has determinant equal to $-\delta_0 \neq 0$ and therefore, by the implicit function theorem, one can solve $\gamma = 0$ for $\lambda = \lambda_0(w)$ and $\epsilon = \epsilon_0(w)$ such that $\lambda_0(0) = \lambda_0$ and $\epsilon_0(0) = 0$.

However, we require that $\epsilon_0(w) > 0$ for definiteness and this can only hold, for small enough $|w|$, if

$$0 > \hat{C}(\lambda_0, 0) \hat{B}(\lambda_0, 0) = u^T C(\lambda_0, 0) B(\lambda_0, 0) k = -\omega_0^2.$$

Using (16), there is a function W such that $W(0) = -\omega_0^2$ and $\epsilon_0(w)w^{-2} + W(w) \rightarrow 0$ as $w \rightarrow 0$. The eigenvalues of (12) are thus given by $\pm \epsilon_0(w)\mu = \pm i\epsilon_0(w)/w$ whose algebraic simplicity follows by the uniqueness from the implicit function theorem. As (15-16) is invariant under $w \mapsto -w$, the functions λ_0 and ϵ_0 must be even. \square

It is precisely the conditions in Theorem 3 which, when applied to the DAE (2), imply the existence of a simple SIB point.

Corollary 1. *Suppose that $\delta_0 \cdot \omega_0 \neq 0$, then there is a function $\tilde{\lambda}(\cdot)$ such that $\tilde{\lambda}(0) = 0$ and the parameterised curve $\{(\lambda(w), \epsilon(w)) : w \in N_\delta(\lambda_0)\}$ is locally the graph $\{(\tilde{\lambda}(\epsilon), \epsilon) : \epsilon \in [0, \epsilon_0]\}$. For $\epsilon > 0$ sufficiently small, the purely imaginary eigenvalue from Theorem 3 can be written as a function of ϵ , namely $i\tilde{w}(\epsilon) \in \sigma(K(\tilde{\lambda}(\epsilon), \epsilon))$ and this function satisfies $\tilde{w}(\epsilon) = |\omega_0|\epsilon^{1/2} + O(\epsilon^{3/2})$ as $\epsilon \rightarrow 0_+$.*

Proof. Denote $\Omega = w^2$ and write equations (15-16) as $\Gamma(\Omega, \lambda, \epsilon) = 0$. Since

$$\det(d_{\lambda, \Omega} \Gamma(0, \lambda_0, 0)) = \begin{vmatrix} -\delta_0 & \hat{C} \hat{A} \hat{B} \\ 0 & -\omega_0^2 \end{vmatrix} = \delta_0 \omega_0^2 \neq 0$$

one can solve equations (15-16) for $\lambda = \tilde{\lambda}(\epsilon), \Omega = \tilde{\Omega}(\epsilon)$ using the implicit function theorem in a neighbourhood of $(0, \lambda_0, 0)$. From the proof of Theorem 3, the purely imaginary eigenvalues of $K(\tilde{\lambda}(\epsilon), \epsilon)$ are $\pm i$ multiplied by

$$\frac{\epsilon}{\tilde{\Omega}(\epsilon)^{1/2}} \stackrel{\text{def}}{=} \tilde{w}(\epsilon)$$

and the result follows using Taylor's theorem. \square

This demonstrates that a simple SIB point in (1) will lead to a curve of Hopf points in (3). It is a natural question to ask whether the same is true of (4) where we have seen that double SIB points occur as λ varies when $\epsilon = 0$.

3.2. Symmetric Slow Subsystems. Let $(x, y) \in \mathbb{R}^{n+m}$. Suppose that (4) has a trivial equilibrium for all (λ, ϵ) and write the linearisation of (4) in the form

$$\mathcal{K}(\lambda, \epsilon) = \frac{1}{\epsilon} \begin{bmatrix} 0 & \epsilon I_n & 0 \\ \epsilon \mathcal{A}(\lambda, \epsilon) & \epsilon \Theta(\lambda, \epsilon) & \epsilon \mathcal{B}(\lambda, \epsilon) \\ \mathcal{C}(\lambda, \epsilon) & 0 & \mathcal{D}(\lambda, \epsilon) \end{bmatrix}.$$

We shall write this matrix as $\frac{1}{\epsilon} \mathcal{K}_1(\lambda, \epsilon)$.

Theorem 4. *Suppose that $\mathbf{N}(\mathcal{D}(\lambda_0, 0)) = \langle k \rangle$ for some non-zero $k \in \mathbb{R}^m$ and $\sigma(\mathcal{D}(\lambda_0, 0))$ contains no other eigenvalues of zero real part. Denote $\mathbf{N}(\mathcal{D}(\lambda_0, 0)^T) = \langle u \rangle$ and suppose $\delta_0 \stackrel{\text{def}}{=} u^T \mathcal{D}'(\lambda_0, 0)k \neq 0$. If $u^T k = 1$ and*

$$u^T \mathcal{C}(\lambda_0, 0) \Theta(\lambda_0, 0) \mathcal{B}(\lambda_0, 0) k = \omega_1^2 > 0$$

then there is a smooth curve $w \mapsto (\lambda_0(w), \epsilon_0(w))$, for $w \in (-\delta, \delta)$, such that

$$\pm i \frac{\epsilon_0(w)}{w} \in \sigma(\mathcal{K}_1(\lambda_0(w), \epsilon_0(w))).$$

Moreover, this represents an algebraically simple eigenvalue. Furthermore,

$$\epsilon_0(0) = 0, \lambda_0(0) = \lambda_0, \lim_{w \rightarrow 0^+} \frac{\epsilon_0(w)}{w^4} = \omega_1^2$$

and both λ_0 and ϵ_0 are even functions.

Proof. As in Theorem 3 one can place the problem on a centre manifold. We have

$$(17) \quad \ddot{x} = \epsilon(\bar{\mathcal{A}}(\lambda, \epsilon)x + \Theta(\lambda, \epsilon)\dot{x} + \bar{\mathcal{B}}(\lambda, \epsilon)\alpha) + h_1(x, \dot{x}, \alpha, \lambda, \epsilon) \in \mathbb{R}^n$$

$$(18) \quad \dot{\alpha} = \bar{\mathcal{C}}(\lambda, \epsilon)x + \bar{\mathcal{D}}(\lambda, \epsilon)\alpha + h_2(x, \dot{x}, \alpha, \lambda, \epsilon) \in \mathbb{R}$$

where we denote

$$\bar{\mathcal{A}}(\lambda, \epsilon) = \mathcal{A}(\lambda, \epsilon) - \mathcal{B}(\lambda, \epsilon)\Delta^{-1}\Pi\mathcal{C}(\lambda_0, 0), \quad \bar{\mathcal{B}}(\lambda, \epsilon) = \mathcal{B}(\lambda, \epsilon)k$$

$$\bar{\mathcal{C}}(\lambda, \epsilon) = u^T [\mathcal{C}(\lambda, \epsilon) - \mathcal{D}(\lambda, \epsilon)\Delta^{-1}\Pi\mathcal{C}(\lambda_0, 0)], \quad \bar{\mathcal{D}}(\lambda, \epsilon) = u^T \mathcal{D}(\lambda, \epsilon)k.$$

Here Θ remains unaffected by the reduction.

In order to find purely imaginary eigenvalues of \mathcal{K}_1 , make the following replacements in (15-16)²

$$\Omega = w^2, \hat{A} \rightsquigarrow \begin{pmatrix} 0 & I \\ \bar{\mathcal{A}} & \Theta \end{pmatrix}, \hat{B} \rightsquigarrow \begin{pmatrix} 0 \\ \bar{\mathcal{B}} \end{pmatrix}, \hat{C} \rightsquigarrow (\bar{\mathcal{C}} \ 0), \delta \rightsquigarrow \bar{D}.$$

²We see, using this notation, that $\hat{C}(\lambda, \epsilon)\hat{B}(\lambda, \epsilon) \equiv 0$. This is why Theorem 3 does not apply directly to equation (4). This property also causes double SIB points when $\epsilon = 0$.

We must now solve

$$(19) \quad 0 = -\delta(\lambda, \epsilon) + \Omega \cdot \hat{C} \hat{A} \hat{B} + O(\Omega^2) = -\delta(\lambda, \epsilon) + \Omega \cdot \bar{C} \bar{B} + O(\Omega^2)$$

$$(20) \quad 0 = \epsilon + \Omega \cdot \hat{C} \hat{B} - \Omega^2 \cdot \hat{C} \hat{A}^2 \hat{B} + O(\Omega^3) = \epsilon - \Omega^2 \cdot \bar{C} \Theta \bar{B} + O(\Omega^3)$$

where the reference to the variables (λ, ϵ) is omitted for clarity. One can solve (19-20) using the implicit function theorem near the point $(\lambda, \epsilon, \Omega) = (\lambda_0, 0, 0)$ for $\lambda = \lambda_0(w)$ and $\epsilon = \epsilon_0(w)$ such that $\epsilon_0(0) = 0$. This gives the existence of the Hopf curve under the further observation that for $\epsilon(w) > 0$ to hold we require $0 < \mathcal{C} \Theta \mathcal{B}(\lambda_0, 0) = \omega_1^2$. It is clear that $\epsilon_0(w)w^{-4} \rightarrow \omega_1^2$ as $w \rightarrow 0$. \square

Corollary 2. *Suppose that $\delta_0 \cdot \omega_1 \neq 0$, then there is a function $\tilde{\lambda}(\cdot)$ such that the parameterised curve $\{(\lambda_0(w), \epsilon_0(w)) : w \in N_\delta(\lambda_0)\}$ is locally the graph $\{(\tilde{\lambda}(\epsilon), \epsilon) : \epsilon \in [0, \epsilon_0]\}$. For $\epsilon > 0$ sufficiently small, the purely imaginary eigenvalue from Theorem 4 can be written as a function of ϵ , namely $i\tilde{w}(\epsilon) \in \sigma(\mathcal{K}_1(\tilde{\lambda}(\epsilon), \epsilon))$ and this function satisfies*

$$\tilde{w}(\epsilon) = |\omega_1|^{1/2} \epsilon^{3/4} + O(\epsilon^{5/4})$$

and

$$\tilde{\lambda}(\epsilon) = \lambda_0 + \epsilon^{1/2} \left(\frac{u^T \mathcal{C}(\lambda_0, 0) \mathcal{B}(\lambda_0, 0) k}{u^T \mathcal{D}'(\lambda_0, 0) k |\omega_1|} \right) + O(\epsilon)$$

as $\epsilon \rightarrow 0_+$.

Proof. Solve (19) for $\lambda = \lambda(\epsilon, \Omega)$ near $(\lambda, \epsilon, \Omega) = (\lambda_0, 0, 0)$ and substitute this into (20) which is then solved for Ω as a function of ϵ using the saddle-node bifurcation theorem. The imaginary part in question is given by $\tilde{w}(\epsilon) = \epsilon / \sqrt{\Omega(\epsilon)}$ and $\tilde{\lambda}(\epsilon) = \lambda(\epsilon, \Omega(\epsilon))$. \square

It follows trivially that

$$\pm i \left[|\omega_1|^{1/2} \epsilon^{-1/4} + O(\epsilon^{1/4}) \right] \in \sigma(\mathcal{K}(\tilde{\lambda}(\epsilon), \epsilon))$$

for all small enough ϵ .

3.3. Conclusions. It appears that one can always solve (15-16) to give a Hopf curve using the implicit function theorem, irrespective of any particular structure present in the mapping $K(\lambda, \epsilon)$. However, this is clearly not the case when the damping term $\Theta(\lambda_0, 0)$ is absent. In this particular setting equation (16) degenerates to ‘ $\epsilon = 0$ ’.

In particular, the system

$$(21) \quad \ddot{x} = f(x, y, \lambda, \epsilon), \quad \epsilon \dot{y} = g(x, y, \lambda, \epsilon)$$

cannot satisfy either Theorem 3 or 4. This is important to those electrical engineers who view a power system as a singularly perturbed ODE which is represented after some transient period by an index-1 DAE. Simply adding

the parasitic $\epsilon \dot{y}$ term to the constraint in DAE models can, of course, drastically change the nature of the dynamics, although this is sometimes performed [10, 11].

For instance, the inclusion of parasitic terms introduces damping into the system. To see this re-scale (21) and write it in the form

$$(22) \quad x_{\tau\tau} = \epsilon^2 f(x, y, \lambda, \epsilon), y_{\tau} = g(x, y, \lambda, \epsilon).$$

One can use the centre manifold theorem to conclude the existence of an invariant, slow manifold [7, 12] given by the graph

$$y = h(x, x_{\tau}, \lambda, \epsilon).$$

The solution manifold of the corresponding ($\epsilon = 0$) DAE subsystem of (22) will be given, locally to some initial conditions, by a velocity-free graph

$$y = y(x, \lambda).$$

This yields a locally conservative reduced-order system. The presence of the damping term x_{τ} in h is induced solely by the parasitic terms $\epsilon \dot{y}$, yet no damping is present in the reduced-order DAE system.

4. A 3-BUS POWER SYSTEM

In this section we take a power system model from [16, 17] and examine some of the dynamics which occur because of the existence of double SIB points within the model. The ‘3-bus power system’ in question is to be found in [17] (p.989) and can be written in the form

$$(23) \quad \begin{aligned} \ddot{\beta} + \cos^2(\alpha) \sin(2\beta) &= \lambda \\ 0 &= 2 \sin(2\alpha) \cos^2(\beta) - 1. \end{aligned}$$

To avoid any confusion we adopt the notation used in [17]. With the damping terms inserted (23) becomes

$$(24) \quad \begin{aligned} \ddot{\beta} + \delta \dot{\beta} + \cos^2(\alpha) \sin(2\beta) &= \lambda \\ \epsilon \dot{\alpha} &= 2 \sin(2\alpha) \cos^2(\beta) - 1. \end{aligned}$$

In both cases the angle variables α, β lie in the unit circle, $S^1 = [0, 2\pi]/(0 = 2\pi)$.

From [17] we have taken the reactive power requirement at the load to be $Q_3 = 0$ and the shunt susceptance is assumed to be $B = 1$. The active power at the load is $P_3 = 1$. From the model, the difference in active power injected into the network by the two buses is given by λ which is written as $\Delta P/2$ in [17]. Following Kwatny we have written α to be twice the difference of the generator and the rest of the network phase angles. Here, β is a combination of the voltage phase angle at the load and of the local generator angle, measured with respect to the rest of the network. For more information on this model [16, 17] should be consulted as there have been several transformations applied to a higher-dimensional 3-bus power system to bring it into this convenient form.

FIGURE 1. Bifurcation diagram with λ versus α . The solid dot is a saddle-node point and the circle is an SIB point.

We begin by finding the steady-states of (23) and therefore of those of (24) too. These are simply the solutions of

$$\cos^2 \beta \sin(2\alpha) = \frac{1}{2}, \quad \sin(2\beta) \cos^2 \alpha = \lambda.$$

This allows use to solve for λ as a function of α and therefore to obtain a plot of the bifurcations of the system as λ changes. We find that

$$(25) \quad \lambda^2 = \frac{2 \sin(2\alpha) - 1}{4 \tan^2 \alpha}$$

for $\pi/12 \leq \alpha \leq 5\pi/12$ and note that λ is zero on the boundary of this interval. A saddle-node bifurcation (SNB) occurs when λ has its maximum value, at which $\frac{d\lambda}{d\alpha} = 0$. This is true if

$$1 + 4 \sin \alpha \cos^3 \alpha - 6 \sin \alpha \cos \alpha = 0$$

and therefore $z = \cos^2 \alpha$, where $1 + z(z - 1)(4z - 6)^2 = 0$. This is depicted in Figure 1 where one can see a pair of saddle-node bifurcation points.

To determine the geometry of the dynamics of (23) we differentiate its constraint to give the singular ODE

$$(26) \quad \begin{aligned} \dot{\beta} &= \xi \\ \dot{\xi} &= \lambda - \cos^2(\alpha) \sin(2\beta) \\ \dot{\alpha} \cos(2\alpha) \cos \beta &= \xi \sin(2\alpha) \sin \beta. \end{aligned}$$

Now re-scale time according to

$$\frac{ds}{dt} = \frac{1}{\cos(2\alpha(t)) \cos \beta(t)}$$

FIGURE 2. Orbits of the DAE power system at $\lambda = 0$ showing two pseudo-equilibria p . The two lines passing through the points p mark the singular manifolds, S . The cross \otimes indicates two stable equilibria.

to give the ODE

$$(27) \quad \begin{aligned} \beta_s &= \xi \cos(2\alpha) \cos \beta \\ \xi_s &= (\lambda - \cos^2(\alpha) \sin(2\beta)) \cos(2\alpha) \cos \beta \\ \alpha_s &= \xi \sin(2\alpha) \sin \beta. \end{aligned}$$

The set

$$C = \left\{ (\alpha, \xi, \beta) \in \mathbb{R}^3 : \cos^2 \beta \sin(2\alpha) = \frac{1}{2} \right\}$$

is called the *constraint* or *solution manifold* for (23). This set is invariant for both the singular ODE (26) and the smooth ODE (27).

Notice that if

$$(28) \quad \cos^2 \beta \sin(2\alpha) = \frac{1}{2}, \quad \cos(2\alpha) \cos \beta = 0 \quad \text{and} \quad \xi = 0$$

then (27) has an equilibrium on the constraint manifold which is not an equilibrium of the DAE (23). Note that the integral curves of (27) do coincide geometrically with those of the DAE (23) on the constraint manifold.

Solving (28) gives the so-called *pseudo-equilibrium* points of (23) and we find $\cos \beta = \pm 1/\sqrt{2}$ and $\cos(2\alpha) = 0$, whence $\beta = \pm\pi/4$ and $\alpha = \pi/4$. Now $\cos^2 \beta \sin(2\alpha)$ is constant on the orbits of (27) and therefore the unstable and stable manifolds of the pseudo equilibria of (27) lie on the solution manifold. It follows that these are also invariant manifolds, or simply *orbits*, of the DAE (23) and determine the nature of the local dynamics of (23) near pseudo-equilibria. One can find further information concerning the dynamics near pseudo-equilibria in [22].

FIGURE 3. The ‘lower half’ of phase space of the DAE power system at $\lambda = 0$, as shown in Figure 2. Here α is plotted against β . This shows the two pseudo-equilibria p and the existence of two ‘pseudo-heteroclinic’ orbits, Γ and its reflection which connect the two pseudo-equilibria. The dotted lines marked S denote the singular manifold and the script S denotes a line of symmetry.

For $\delta \neq 0$ a double SIB point occurs in (23) when an equilibrium encounters the *singular manifold* given by

$$S = \left\{ (\alpha, \xi, \beta) \in \mathbb{R}^3 : \cos^2 \beta \sin(2\alpha) = \frac{1}{2}, \cos(2\alpha) \cos^2 \beta = 0 \right\}.$$

Hence all pseudo-equilibria lie in the singular manifold. Notice that the singular manifold is simply the set of points in the constraint manifold where one cannot use the implicit function theorem directly to solve the constraint for α as a function of β .

Theorem 1 tells us that the hyperbolicity of eigenvalues is lost at a double SIB point λ_0 because they behave as $\pm O(\lambda - \lambda_0)^{-1/2}$. In the power system model (23) there is a reversible SIB point at $\lambda_0 = \pm 1/2$. This is found by substituting $\alpha = \pi/4$ into the expression for λ^2 given by (25).

Using this information we can draw the integral curves of (23) for different values of λ , as shown in Figures 2 and 4. The horizontal lines in Figure 2 passing through the two pseudo equilibrium points marked ‘p’ contain all the impasse points for (23). For background information concerning the dynamics near impasse points see [9, 18].

One additional point to note is that the closed curve marked Γ in Figures 2 and 3 appears to form heteroclinic connections between the two pseudo-equilibria ‘p’. In fact, Γ forms two distinct periodic orbits because the arrival time at the pseudo-equilibria along their invariant manifolds is *finite*.

To see this argue as follows. Let $p = (\pi/4, 0, \pi/4)$ be one of the pseudo-equilibria and note that in s -time along the heteroclinic orbit, $W^s(p)$, we have $\alpha(s) = \pi/4 + O(e^{-ls})$ as $s \rightarrow \infty$. Here $-l$ is the negative eigenvalue

FIGURE 4. Phase space of the DAE power system where λ lies between the SIB point and the SNB point. The cross \otimes marks the two equilibria, one of which is stable and one is a saddle. There is a homoclinic orbit connecting the saddle point to itself. Notice that one of the pseudo-equilibria marked p is a pseudo-centre. See Figure 5 for a schematic of the ‘inner part’ of this diagram.

missing figure

associated with the linearisation of (23) at the pseudo-equilibrium. Therefore $\cos(2\alpha(s)) = \sin O(e^{-ls}) = O(e^{-ls})$ for large $s > 0$. Using the change in time-scale, we know that if the orbit of some solution, $\cup_s(\alpha(s), \dot{\beta}(s), \beta(s))$ say, forms the connecting orbit $W^s(p)$, then the arrival time at p from some nearby point of the DAE (23) along this manifold is given by

$$T(s_0) = T = \int_0^T dt = \int_{s_0}^{\infty} \frac{dt}{ds} ds = \int_{s_0}^{\infty} \cos(2\alpha(s)) \cos(\beta(s)) ds.$$

Therefore

$$|T| \leq \int_{s_0}^{\infty} |\cos(2\alpha(s)) \cos(\beta(s))| ds \leq \int_{s_0}^{\infty} O(e^{-ls}) ds$$

which is clearly finite. Here, s_0 is any large enough initial s -time. It follows that the heteroclinic orbits of (27) in s -time become periodic orbits of (23) in t -time which connect one component of $C \setminus S$ to another.

This shows us that the stable and unstable manifolds of the pseudo-equilibria in the rescaled time-scale denoted by the variable s become two distinct periodic orbits for the DAE in the t time-scale. This behaviour arises because there need not be any uniqueness of solutions of (23) along the singular manifold.

In Figure 4 there are shown two equilibria which lie on the same component of the solution manifold $C \setminus S$ and there is a homoclinic orbit on this component. This is illustrated in Figure 5. The homoclinic orbit comes into existence at the same point as one of the equilibria passes the singular manifold: the SIB point. When the equilibria are on different components of the solution manifold there are no homoclinic or heteroclinic connections between them.

In terms of the damped power system (24) we may apply Corollary 2 to conclude that the SIB point which occurs in (23) at $\lambda = \pm 1/2$ implies that for all $\delta \neq 0$ and sufficiently small $\epsilon > 0$, there are two curves of Hopf points in (λ, ϵ) -space given by

$$\tilde{\lambda}^{\pm}(\epsilon) = \pm \frac{1}{2} \left(1 - \frac{\epsilon^{1/2}}{\sqrt{2\delta}} + O(\epsilon) \right).$$

FIGURE 5. The ‘inner-part’ of phase space of the DAE power system near the SIB point from Figure 4. Here α is plotted against β . This shows the pseudo-equilibrium p and the existence of a homoclinic orbit Γ_1 . S denotes the singular manifold and the script S denotes a line of symmetry. The curve Γ_2 is an *equilibrium-singularity* connection, Γ_3 reverses the orientation of Γ_2 and Γ_4 is a *singularity-singularity* connection. There is a continuum of these because p is a *pseudo-centre*.

On this curve the eigenvalues of the linearisation of (24) at its equilibrium are given by

$$\pm i\tilde{\omega}(\epsilon) = \pm i[(2\delta)^{1/4}\epsilon^{-1/4} + O(\epsilon^{1/4})].$$

Two of the resulting Hopf bifurcations are shown in Figure 6 and they are seen to be subcritical. Computations done in AUTO indicate that the branch of periodic orbits terminates in a homoclinic orbit connecting the unstable equilibrium to itself.

In Figure 7 one can see a particular periodic solution near to the end of the branch where $(\lambda, \|\alpha\|) \simeq (-0.3, 1.27)$. It shows a solution which moves rapidly from the unstable part to the stable part of the slow manifold. The ‘diagonal’ part of this curve lies on the folded slow-manifold and appears to pass through the fold at the pseudo-equilibrium which is indicated by the solid dot at $(\beta, \dot{\beta}, \alpha) = (\pi/4, 0, \pi/4)$.

REFERENCES

- [1] A. Ambrosetti and G. Prodi. *A Primer of Nonlinear Analysis*, volume 34 of *Cambridge studies in Advanced Mathematics*. Cambridge University Press, 1992.
- [2] V. I. Arnold, V. S. Afrajmovich, Yu. S. Il'yashenko, and L. P. Shil'nikov. *Bifurcation Theory and Catastrophe Theory*. Springer, 1999.
- [3] S. M. Baer and T. Erneux. Singular Hopf bifurcation to relaxation oscillations. *SIAM Appl. Math.*, 52(6):1651–1664, 1986.
- [4] R. E. Beardmore. Stability and bifurcation properties of index-1 DAEs. *Numerical Algorithms*, 19(1–4):43–53, December 1998.

FIGURE 6. A computed curve of periodic solutions bifurcating from the equilibrium locus; $\delta = 1$ and $\epsilon = 1/100$. Here, λ is plotted against $\|\alpha\| = \max_t |\alpha_\lambda(t)|$ on each branch and a subcritical Hopf bifurcation occurs $O(\epsilon^{1/2})$ -near to the SIB point $\lambda = \pm 1/2$. The dotted lines represent unstable equilibria and limit cycles whereas thick lines denote stable equilibria. This has been computed using AUTO.

FIGURE 7. A plot of an orbit from the bifurcating branch of periodic orbits in Figure 6. This orbit is found near to the end of the branch. The manifolds $M^{s,u}$ are the stable and unstable parts of the slow manifold. This solution has been computed using AUTO and it appears to pass through the pseudo-equilibrium point situated at $(\beta, \dot{\beta}, \alpha) = (\pi/4, 0, \pi/4)$.

[5] R. E. Beardmore. The singularity-induced bifurcation and its Kronecker normal form. to appear, July 2000.

- [6] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. North Holland, Elsevier Science Publishing, 1989.
- [7] J. Carr. *Applications of Centre Manifold Theory*. Springer-Verlag, 1981.
- [8] Hsiao-Dong Chiang, Chia-Chi Chu, and Gerry Cauley. Direct stability analysis of electrical power systems using energy functions: Theory, applications and perspective. *Proceedings of the IEEE*, 83(11):1497–1527, 1995.
- [9] L. O. Chua. Impasse points. part II. analytical aspects. *Int. J. of Circuit Theory and Applications*, 17:271–282, 1989.
- [10] C. L. DeMarco and A. R. Bergen. A security measure for random load disturbances in nonlinear power system models. *IEEE CAS Part I*, 34(12):1546–1557, 1987.
- [11] I. Dobson and H. D. Chiang. Towards a theory of voltage collapse in electric power systems. *Systems and Control Letters*, 13:253–262, 1989.
- [12] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *Differential Equations*, 31:53–98, 1979.
- [13] F. R. Gantmacher. *Theory of Matrices*. Chelsea Publishing Co., New York, 1977.
- [14] IEE. *Colloquium on Voltage Collapse*, Savoy Place, London, WC2R OBL., April 1997.
- [15] N. Kopell and R. B. Washburn Jr. Chaotic motion in the two-degree-of-freedom swing equations. *IEEE CAS Part I*, 29(11):738–746, 1982.
- [16] H. G. Kwatny, R. F. Fischl, and C. O. Nwankpa. Local bifurcation in power systems: Theory, computation and application. *Proceedings of the IEEE*, 83(11), November 1995.
- [17] H. G. Kwatny, A. K. Pasrija, and L. Y. Bahar. Static bifurcations in electric power networks: Loss of steady-state stability and voltage collapse. *IEEE Transactions on Circuits and Systems*, 33(10):981–991, October 1986.
- [18] P. J. Rabier and W. C. Rheinboldt. On impasse points of quasilinear differential-algebraic equations. *Math. Anal. and Appns.*, 181:429–454, 1994.
- [19] F. M. A. Salam, J. E. Marsden, and P. P. Varaiya. Arnold diffusion in the swing equations of a power system. *IEEE CAS Part I*, 31(8):673–688, 1984.
- [20] S. Sastry and P. Varaiya. Hierarchical stability and alert state steering control of interconnected power systems. *IEEE CAS Part I*, 27(11):1102–1112, 1980.
- [21] M. Stiefenhofer. Singular perturbation with limit points in the fast dynamics. *ZAMP*, 49(65):730–758, 1998.
- [22] V. Venkatasubramanian, H. Schattler, and J. Zaborszky. A stability theory of large differential algebraic systems, a taxonomy. No SSM 9201 Part 1. Technical report, Washington University, Washington, USA, 1992.
- [23] A. K. Zvonkin and M. A. Shubin. Non-standard analysis and singular perturbations of ordinary differential equations. *Russian Mathematical Surveys*, 39(2):69–131, 1984.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, SOUTH KENSINGTON CAMPUS,,
UNIVERSITY OF LONDON, LONDON, SW7 2AZ.