SIAM J. MATRIX ANAL. APPL.
Vol. 24, No. 1, pp. 106–120

THE FLOW OF A DAE NEAR A SINGULAR EQUILIBRIUM*

R. E. BEARDMORE† AND R. LAISTER‡

Abstract. We extend the *differential-algebraic equation (DAE) taxonomy* by assuming that the linearization of a DAE about a singular equilibrium has a particular index-2 Kronecker normal form. A Lyapunov–Schmidt procedure is used to reduce the DAE to a quasilinear normal form which is shown to posses quasi-invariant manifolds which intersect the singularity. In turn, this provides solutions of the DAE which pass through the singularity.

Key words. Lipschitz solutions, nonhyperbolic equilibrium, quasi-invariant manifolds

AMS subject classifications. 34A34, 34A09, 37G05

PII. S0895479800378660

1. Preliminaries.

We consider the differential-algebraic equation (DAE)

$$\dot{x} = f(x, y), \tag{1.1}$$

$$g(x, y) = 0, \tag{1.2}$$

where $x \in \mathbb{R}^n$ $(n \geq 2)$, $y \in \mathbb{R}^m$, and $f : \mathcal{U} \to \mathbb{R}^n$ and $g : \mathcal{U} \to \mathbb{R}^m$ are both C^ω (analytic) in an open neighborhood, $\mathcal{U}$, of $(0, 0)$ in $\mathbb{R}^{n+m}$. The motivation for this paper is to understand the orbit structure of (1.1)–(1.2) near $(0, 0)$, which is assumed to be a *singular equilibrium* in the sense that

A1. $f(0, 0) = 0,\ g(0, 0) = 0$,
A2. $\mathrm{N}(d_y g(0, 0)) = \langle k \rangle, k^T k = 1$, where $\mathrm{N}(d_y g(0, 0)^T) = \langle u \rangle$.

We shall also make the following assumptions, which we introduce now in order to make the presentation as transparent as possible:

A3. $d_x g(0, 0) d_y f(0, 0) k \notin \mathrm{R}(d_y g(0, 0))$,
A4. $d(f \times g)(0, 0) \in GL(\mathbb{R}^{n+m})$, and
A5. $d^2_{yy} g(0, 0)[k, k] \notin \mathrm{R}(d_y g(0, 0))$.

There is one further condition to be imposed which will be introduced at the appropriate point in the paper. The regularity assumptions are imposed on f and g for brevity, and one could consider problems of finite smoothness in a similar manner.

First, let us define some terminology associated with (1.1)–(1.2). The *constraint manifold* for (1.1)–(1.2) is the set $\mathbf{C} = \{(x, y) \in \mathcal{U} : g(x, y) = 0\}$, and the *singularity* is $\mathbf{S} = \{(x, y) \in \mathbf{C} : \det(d_y g(x, y)) = 0\}$.

The main result of the paper is that one can use A1–A5 to reduce the DAE (1.1)–(1.2) to a *quasilinear normal form* of dimension n. This normal form is a differential equation which can be written as

$$\dot{\alpha} = L_0 \alpha + \mathcal{O}(2), \tag{1.3}$$

$$s(\alpha, \beta)\dot{\beta} = \beta + \mathcal{O}(2), \tag{1.4}$$

*Received by the editors September 20, 2000; accepted for publication (in revised form) by V. Mehrmann January 16, 2002; published electronically June 12, 2002.
http://www.siam.org/journals/simax/24-1/37866.html

†Department of Mathematics, Imperial College, South Kensington, University of London, London, England, SW7 2AZ (r.beardmore@ic.ac.uk).
‡School of Mathematical Sciences, University of the West of England, Frenchay Campus, Bristol, England, BS16 1QY (robert.laister@uwe.ac.uk).

where $(\alpha, \beta) \in \mathbb{R}^n, L_0 \in GL(\mathbb{R}^{n-1})$ is some mapping and $s(0,0) = 0$. We can then understand the nature of solutions of (1.3)–(1.4), and hence of the original DAE, by rescaling time and applying standard invariant manifold theory to the resulting ODE. The only proviso to be met in this process is that solutions of (1.3)–(1.4) will require a degree of differentiability that is not imposed by the formulation (1.1)–(1.2).

1.1. Background. A standard uniqueness theorem for differential equations implies that for any $(x_0, y_0) \in \mathbf{C} \backslash \mathbf{S}$ there exist $\alpha, \omega > 0$ and a unique C^ω solution of (1.1)–(1.2), $(-\alpha, \omega) \to \mathbb{R}^{n+m}$; $t \mapsto (x(t), y(t)) \in \mathbf{C} \backslash \mathbf{S}$, such that $(x(0), y(0)) = (x_0, y_0)$. The goal of this paper is therefore to try to understand the nature of solutions which encounter the singularity and to understand how uniqueness can break down.

The usual alternative for the global continuation of solutions of ODEs states that solutions either exist for all time or else become unbounded in finite time. There is a third alternative for solutions of DAEs: the solutions terminate at a singularity [8]. However, it is not true that all solutions which encounter the singularity must terminate there; some may be continued [11, 12]. Indeed, the *DAE taxonomy* described in these references gives conditions under which there are submanifolds of $\mathbf{S}$ where such a continuation is possible.

In [3], the authors discuss the possibility of using the *DAE taxonomy* to investigate a type of shock wave in a magneto-hydrodynamics equation which makes this paper also relevant to that study. In [7] März gives conditions to ensure that the semilinear DAE

$$\mathcal{A}\dot{z} + \mathcal{B}z = \varphi(z), \qquad \|\varphi(z)\| = O(\|z\|^2) \text{ as } z \to 0, \tag{1.5}$$

has a Lyapunov stable equilibrium. In particular, the author supposes that the *Kronecker index* of the matrix pencil $(\mathcal{A}, \mathcal{B})$ is two, and in due course we shall write (1.1)–(1.2) in this form.

1.2. Notation. The term *manifold* is taken as a synonym for *graph* and the tangent space of a manifold $\mathcal{M}$ at a point $z \in \mathcal{M}$ is written $T_z(\mathcal{M})$. If U is a linear space, then for each $u \in U$ we shall write the map $v \mapsto u^T v$ as u^T, and the span of u is written as $\langle u \rangle = \{\mu u : \mu \in \mathbb{R}\}$. Also, $\|u\|^2 := u^T u$ and a hash symbol (#) denotes set cardinality.

Let $(\mathcal{A}, \mathcal{B}) \in \mathcal{L}(\mathbb{R}^N) \times \mathcal{L}(\mathbb{R}^N)$ be a square matrix pencil. It is *regular* if there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda\mathcal{A} + \mathcal{B}) \neq 0$. The spectrum of $(\mathcal{A}, \mathcal{B})$ is $\sigma(\mathcal{A}, \mathcal{B}) := \{\lambda \in \mathbb{C} : \det(\lambda\mathcal{A} + \mathcal{B}) = 0\}$, and $(\mathcal{A}, \mathcal{B})$ is *hyperbolic* if $\sigma(\mathcal{A}, \mathcal{B})$ contains no purely imaginary elements. We write $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, and $\mathbb{C}^-$ is defined similarly.

Let us stipulate the degree of smoothness of solutions of (1.1)–(1.2) as follows. If $I \subset \mathbb{R}$ is open, a solution of (1.1)–(1.2) is a map $t \mapsto (x(t), y(t)) \in C^1(I, \mathbb{R}^n) \times C^0(I, \mathbb{R}^m)$, such that (1.1)–(1.2) is satisfied for all $t \in I$. A set $K \subset \mathbf{C}$ is said to be *quasi-invariant* for (1.1)–(1.2) if for each $(x(0), y(0)) \in K$ there is *at least one* solution of (1.1)–(1.2), $(x, y) : I \to \mathbf{C}$, such that $(x(t), y(t)) \in K$ for all $t \in I$.

In order to discuss solutions of the quasilinear problem (1.3)–(1.4), we must impose some degree of differentiability. So, let $I \subset \mathbb{R}$ be a bounded interval and let us note at this stage that the setting for solutions of (1.3)–(1.4) will be the space of Lipschitz functions. Thus, let us denote the Sobolev space

$$W^{n,\infty}(I, \mathbb{R}) = \left\{ \beta : I \to \mathbb{R} : \beta, \dot{\beta}, \ldots, \beta^{(n)} \in L^\infty(I) \right\},$$

endowed with the standard norm, $\|u\|_{W^{n,\infty}} = \sum_{j=0}^{n} \|u^{(j)}\|_{L^\infty}$, where a dot and superscript (j) represent the derivative in a weak sense. Due to the inequality

$$|u(x) - u(y)| \le |x-y| \|u\|_{W^{1,\infty}} \qquad \forall x, y \in I,$$

we may consider elements of $W^{n,\infty}(I, \mathbb{R})$ as being those functions with a Lipschitz continuous nth derivative.

1.3. A Kronecker normal form. In [2] there is a Kronecker normal form (KNF) which will provide the basis for the construction of the quasilinear normal form (1.3)–(1.4). First, let us define the matrices

$$M := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+m}).$$

We then have the following result from [2] concerning the KNF of (M, L).

THEOREM 1.1. *Suppose that $n \ge 2$ and $\det L \neq 0$. If $\mathrm{N}(D) = \langle k \rangle$ for some nonzero $k \in \mathbb{R}^m$ such that $CBk \notin \mathrm{R}(D)$, then there are nonsingular transformations P and Q such that*

$$PMQ = \begin{bmatrix} I_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_0 & 0 \end{bmatrix} \quad \text{and} \quad PLQ = \begin{bmatrix} A_0^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

where $C_0 : \mathbb{R} \to \mathbb{R}^m$ is a linear map such that $C_0(1) = k$. If we write $\mathrm{N}(D^T) = \langle u \rangle$ and $U = \langle C^T u \rangle^\perp$, then $A_0 \in GL(U)$ and $\sigma(M, L) = 1/\sigma(A_0)$, where both PMQ and PLQ are elements of $\mathcal{L}(U \oplus \mathbb{R} \oplus \mathbb{R}^m)$.

If one assumes A1–A6, it follows from Theorem 1.1 that the linear DAE obtained from linearizing (1.1)–(1.2) at the zero equilibrium has index 2.

1.4. An underlying vector field. By writing $z = (x, y) \in \mathbb{R}^{n+m}$ and setting

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} d_x f(0,0) & d_y f(0,0) \\ d_x g(0,0) & d_y g(0,0) \end{bmatrix}, \tag{1.6}$$

we may write (1.1)–(1.2) as the semilinear problem

$$M\dot{z} - Lz = F(z), \tag{1.7}$$

where the C^ω mapping F is defined by $Lz + F(z) = (f \times g)(z)$ and $F(z)$ is $\mathcal{O}(2)$ at zero.

Now consider (1.2) along a solution of (1.1)–(1.2) which lies in $\mathbf{C} \backslash \mathbf{S}$. Differentiating this constraint with respect to time we find

$$\dot{y} = d_y g(x, y)^{-1} d_x g(x, y) f(x, y).$$

By defining the variable τ by

$$\frac{d\tau}{dt} = \frac{1}{\det\, d_y g(x(t), y(t))}, \qquad \tau(t_0) = \tau_0,$$

we can reduce (1.1)–(1.2) to a vector field in the new time-scale τ:

$$x' = f(x, y) \det(d_y g(x, y)), \tag{1.8}$$

$$y' = \mathrm{adj}(d_y g(x, y)) d_x g(x, y) f(x, y), \tag{1.9}$$

where a prime ($'$) denotes $\frac{d}{d\tau}$.

This procedure gives a smooth vector field for which **C** is an invariant manifold, and any invariant set of (1.8)–(1.9) in **C** is a quasi-invariant set for (1.1)–(1.2). Moreover, the orbits of (1.8)–(1.9) coincide geometrically with those of (1.1)–(1.2), and this allows us to infer the behavior of (1.1)–(1.2), even at the singularity. This approach is used in [11] as the basis for the *DAE taxonomy*.

This approach can be useful, as in the following result which shows that when orbits of (1.8)–(1.9) are transverse to **S** at some point, that singular point is an impasse point. First, let us define

$$\Delta(x,y) := \det(d_y g(x,y)).$$

PROPOSITION 1.2. *Suppose that* $\tau \mapsto (x(\tau), y(\tau))$ *is a solution of* (1.8)–(1.9) *with initial condition* $(x(0), y(0)) = (x_0, y_0) \in \mathbf{S}$. *If*

$$\frac{d}{d\tau}\Delta(x(\tau), y(\tau))\bigg|_{\tau=0} \neq 0,$$

then there is a $t_* \in \mathbb{R}$ *such that* (1.1)–(1.2) *has exactly two solutions,* $(x(t), y(t))$, *which are both defined on either* $[t_*, t_* + T)$ *or* $(t_* - T, t_*]$ *for some* $T > 0$ *and which satisfy* $(x(t_*), y(t_*)) = (x_0, y_0)$. *Moreover,* $\|\dot{y}(t)\| \to \infty$ *as* $t \to t_*$.

Proof. From Theorem 2.1 of [9], we have to show that there is some nonzero $k \in \mathbb{R}^m$ such that $\mathrm{N}(d_y g(x_0, y_0)) = \langle k \rangle$, $d_x g(x_0, y_0) f(x_0, y_0) \notin \mathrm{R}(d_y g(x_0, y_0))$ and $d^2_{yy} g(x_0, y_0)[k, k] \notin \mathrm{R}(d_y g(x_0, y_0))$.

Define $\delta(\tau) := \Delta(x(\tau), y(\tau))$, so that $\delta(0) = 0$. Differentiating we have

$$\begin{aligned}\delta'(\tau) &= \frac{d}{d\tau}\Delta(x(\tau), y(\tau)) \\ &= d_x\Delta(x(\tau), y(\tau))x'(\tau) + d_y\Delta(x(\tau), y(\tau))y'(\tau) \\ &= -d_x\Delta \cdot \Delta \cdot f + d_y\Delta \cdot (\mathrm{adj}\ d_y g) \cdot d_x g \cdot f.\end{aligned}$$

Therefore $\delta'(0) = d_y\Delta\,(\mathrm{adj}\ d_y g)\, d_x g\ f|_{(x_0,y_0)}$, which is nonzero by assumption. Since the dimension of $\mathrm{N}(d_y g(x_0, y_0))$ is greater than or equal to two if and only if the adjugate $\mathrm{adj}(d_y g(x_0, y_0))$ is the zero mapping, we have $\delta'(0) = 0$ if $\dim \mathrm{N}(d_y g(x_0, y_0)) \geq 2$. Therefore $\mathrm{N}(d_y g(x_0, y_0)) = \langle \kappa \rangle$ for a nonzero $\kappa \in \mathbb{R}^m$. Now apply Lemma 3 from [1] to deduce that $\mathrm{R}(\mathrm{adj}\ d_y g(x_0, y_0)) = \langle \kappa \rangle$ and $\mathrm{N}(\mathrm{adj}\ d_y g(x_0, y_0)) = \mathrm{R}(d_y g(x_0, y_0))$. Using Lemma 1 from [1] we have

$$d_y\Delta(x,y)[\cdot] = \det{}'(d_y g)\left[d^2_{yy} g(x,y)[\cdot]\right] = \mathrm{tr}\left((\mathrm{adj}\ d_y g) d^2_{yy} g(x,y)[\cdot]\right) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}),$$

where $\det'$ is the derivative of the determinant. Hence

$$\delta'(0) = \mathrm{tr}\left((\mathrm{adj}\ d_y g) d^2_{yy} g\left[(\mathrm{adj}\ d_y g) d_x g\ f(x_0, y_0)\right]\right).$$

We now use the simple null-space of the derivative $d_y g$ to conclude that if

$$d_x g\ [f(x_0, y_0)] \in \mathrm{R}(d_y g(x_0, y_0)), \tag{1.10}$$

then $\delta'(0) = 0$, and (1.10) cannot be true. It follows that there is a *nonzero* l_0 such that $(\mathrm{adj}\ d_y g) d_x g\ f(x_0, y_0) = l_0 \kappa$. Therefore $\delta'(0) = l_0\ \mathrm{tr}\left((\mathrm{adj}\ d_y g) d^2_{yy} g(x_0, y_0)[\kappa]\right)$. Now define the linear mapping

$$T := (\mathrm{adj}\ d_y g) d^2_{yy} g(x_0, y_0)[\kappa];$$

then $\mathrm{R}(T) \subset \langle\kappa\rangle$ and $Ty \equiv \kappa\ell^T y$ for some $\ell \in \mathbb{R}^m$. Hence $\sigma(T) = \{0, \ell^T\kappa\}$ so that $\ell^T\kappa = \mathrm{tr}(T)$. Using Lemma 3 from [1] again, if $d^2_{yy}g(x_0, y_0)[\kappa, \kappa] \notin \mathrm{R}(d_y g(x_0, y_0))$, then $T\kappa \neq 0$ from where $\delta'(0) \neq 0$, and the result follows. □

Generally, $y(t)$ has the form

$$y(t) = O(t - T_*)^{1/2}$$

as $t \to T_*$ at an impasse point. In the degenerate diffusion literature, solutions which have this form, where t represents a spatial variable, are said to be *sharp solutions* [10].

2. A quasilinear normal form. The principle tool in our approach to understanding the flow of (1.1)–(1.2) is given in this section and is based on the following idea. Rather than differentiating the constraint (1.2) to obtain a vector field, suppose instead that we eliminate (1.2) directly by applying the implicit function theorem. Clearly, one cannot solve the constraint for y as a function of x near $(0, 0)$, but since $dg(0, 0)$ has full rank then $\mathbf{C} = g^{-1}(0)$ is a manifold and the information contained in (1.1) will define trajectories on it. However, the way in which the implicit function theorem is used is crucial, and the location of the singularity must emerge from this process. If we choose the correct decomposition of the ambient space in order to apply this Lyapunov–Schmidt reduction, then we can limit the way in which the singularity appears in the reduced problem.

In fact, Theorem 1.1 gives a decomposition through which we can track the effect of the singularity on solutions, and this in turn will allow us to find solutions which are unaffected by the presence of the singularity.

First we prove a preliminary lemma.

LEMMA 2.1. *Suppose that* A1–A5 *hold; then* $\mathbf{C}$ *is a manifold of dimension* n, *and* $\mathbf{S}$ *is a codimension*-1 *submanifold of* $\mathbf{C}$.

Proof. Let $\mathrm{N}(D^T) = \langle u\rangle$ for some nonzero $u \in \mathbb{R}^m$ and note that $C^T u \neq 0$ by A4; recall from A2 that $\mathrm{N}(D) = \langle k\rangle$. Write $y = \alpha k + \kappa \in \langle k\rangle \oplus \langle k\rangle^\perp = \mathbb{R}^m$ and form the decomposition $\mathbb{R}^m = \langle u\rangle \oplus \langle u\rangle^\perp$. Let $P : \mathbb{R}^m \to \langle u\rangle$ and $I - P : \mathbb{R}^m \to \langle u\rangle^\perp$ be orthogonal projections, and write $x = \lambda C^T u + \xi \in \langle C^T u\rangle \oplus \langle C^T u\rangle^\perp$.

Then $g(x, y) = 0 \in \mathbb{R}^m$ if and only if $(I - P + P)g(x, y) = 0$, which suggests that we define the mapping $\Gamma : \mathbb{R} \times \langle k\rangle^\perp \times \mathbb{R} \times \langle C^T u\rangle^\perp \to \langle u\rangle^\perp \times \mathbb{R}$ by

$$\Gamma(\alpha, \kappa, \lambda, \xi) := \begin{bmatrix} (I - P)g(\lambda C^T u + \xi, \alpha k + \kappa) \\ u^T g(\lambda C^T u + \xi, \alpha k + \kappa) \end{bmatrix}.$$

Now

$$d_{\kappa,\lambda}\Gamma(0, 0, 0) = \left[\begin{array}{c|c} (I - P)D|_{\langle k\rangle^\perp} & (I - P)CC^T u \\ \hline u^T D|_{\langle k\rangle^\perp} & u^T CC^T u \end{array}\right] = \begin{bmatrix} (I - P)D|_{\langle k\rangle^\perp} & * \\ 0 & \|C^T u\|^2 \end{bmatrix},$$

where $(I - P)D|_{\langle k\rangle^\perp}$ is a bijection. Hence one can apply the implicit function theorem to solve $g(\lambda C^T u + \xi, \alpha k + \kappa) = 0$ for $\kappa = \kappa(\alpha, \xi)$ and $\lambda = \lambda(\alpha, \xi)$ in a neighborhood of the origin of $\mathbb{R}^{n+m}$.

To locate $\mathbf{S}$ we must solve $g(x, y) = 0$, $\det(d_y g(x, y)) = 0$, and these are satisfied in some neighborhood of the origin if and only if

$$\hat{g}(\alpha, \xi) := \det\left(d_y g(\lambda(\alpha, \xi)C^T u + \xi, \alpha k + \kappa(\alpha, \xi))\right) = 0. \tag{2.1}$$

Now $\hat{g}$ is C^ω and $\hat{g}(0,0) = 0$, so that by Lemma 1 of [1], using the fact that $d_\alpha \lambda(0,0) = 0$ and $d_\alpha \kappa(0,0) = 0$, we have $d_\alpha \hat{g}(0,0) = \mathrm{tr}((\mathrm{adj}D) d^2_{yy} g(0,0)[k])$. Using Lemma 3 of [1] we have $\mathrm{R}(\mathrm{adj}D) = \langle k \rangle$, so that $\mathrm{tr}((\mathrm{adj}D) d^2_{yy} g(0,0)[k])$ coincides with the only nonzero element of $\sigma((\mathrm{adj}D) d^2_{yy} g(0,0)[k])$. But $d^2_{yy} g(0,0)[k,k] \notin \mathrm{R}(D) = \mathrm{N}(\mathrm{adj}D)$ and therefore $(\mathrm{adj}D) d^2_{yy} g(0,0)[k,k] = \eta k$ for some $\eta \neq 0$. Because $d_\alpha \hat{g}(0,0) = \eta$, we may locally solve $\hat{g} = 0$ for $\alpha = \alpha(\xi)$ by the implicit function theorem. ∎

2.1. The main result. From the following result we can deduce many properties concerning the flow of (1.1)–(1.2).

THEOREM 2.2. *Assume* A1–A5 *hold and recall* $U = \left\langle C^T u \right\rangle^\perp \subset \mathbb{R}^n$. *There is a* C^ω*-diffeomorphism* $\chi : B(0,0) \subset U \times \mathbb{R} \to \mathbf{C}$, *where* $B(0,0)$ *is a neighborhood of* $(0,0)$, *with the following properties. The map* $(x(\cdot), y(\cdot))$ *is a solution of* (1.1)–(1.2) *in* $\mathcal{U}$ *with* $k^T y(\cdot) \in W^{1,\infty}(I, \mathbb{R})$ *if and only if* $(x(t), y(t)) = \chi(\alpha(t), \beta(t))$, *where* (α, β) *satisfies*

$$\dot{\alpha} = L_0 \alpha + \rho_0(\alpha, \beta), \tag{2.2}$$

$$s(\alpha, \beta)\dot{\beta} = \beta + \rho_1(\alpha, \beta), \tag{2.3}$$

with $(\alpha, \beta) \in C^1(I, U) \times W^{1,\infty}(I, \mathbb{R})$ *and* (2.2)–(2.3) *satisfied for a.e.* $t \in I$.

The map $L_0 \in GL(U)$ *satisfies* $\sigma(L_0) = \sigma(M, L)$ *and* $\rho_0 \times \rho_1 : B(0,0) \to U \times \mathbb{R}$ *is* C^ω *and* $\mathcal{O}(2)$ *at zero. Moreover,* $s : B(0,0) \to \mathbb{R}$ *is* C^ω *and* $\chi(s^{-1}(0) \cap B(0,0)) = \mathbf{S}$, $s(0,0) = 0$, *and* $d_\beta s(0,0) \neq 0$. *Consequently,* $\Sigma := s^{-1}(0) \subset U \times \mathbb{R}$ *is an* $(n-1)$*-dimensional manifold.*

Proof. Using Theorem 1.1 we may write $\mathbb{R}^n = U \oplus \langle Bk \rangle$ and $\mathbb{R}^m = \langle k \rangle \oplus \langle k \rangle^\perp$. Now write $x = x_0 + x_1 Bk \in U \oplus \langle Bk \rangle$ and $y = y_1 k + y_0 \in \langle k \rangle \oplus \langle k \rangle^\perp$.

As in (1.7), we can write (1.1)–(1.2) as

$$\begin{aligned} \dot{x} &= Ax + By + \mathcal{F}(x,y), \\ 0 &= Cx + Dy + \mathcal{G}(x,y), \end{aligned} \tag{2.4}$$

where $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}(2)$ at $(0,0)$. Hence, the constraint (1.2) becomes

$$\begin{aligned} g(x,y) &= g(x_1 Bk + x_0, y_1 k + y_0) \\ &= x_1 CBk + Cx_0 + Dy_0 + \mathcal{G}(x_1 Bk + x_0, y_1 k + y_0) \\ &=: \Gamma(x_1, x_0, y_1, y_0) \\ &= 0. \end{aligned}$$

Now define the linear mapping $\Delta \in \mathcal{L}(\mathbb{R} \times \langle k \rangle^\perp, \mathbb{R}^m)$ by

$$\Delta[a,b] := d_{(x_1,y_0)} \Gamma(\mathbf{0})[a,b] = aCBk + D|_{\langle k \rangle^\perp} b$$

for $a \in \mathbb{R}$ and $b \in \langle k \rangle^\perp$. Since $\langle u \rangle = \mathrm{N}(D^T)$, then $\Delta[a,b] = 0$ implies $au^T CBk = 0$ so that $a = 0$. Since $Db = 0$ therefore follows and because b lies in a space complementary to $\langle k \rangle$, we find that $b = 0$ too. Since Δ is thus an injection of finite-dimensional spaces of the same dimension, it is a bijection. One can therefore solve $g(x,y) = 0$ locally and uniquely for C^ω functions X and Y such that $x_1 = X(x_0, y_1)$ and $y_0 = Y(x_0, y_1)$.

Now define the local diffeomorphism $\bar{\chi} \in C^\omega(U \times \mathbb{R}, \mathbf{C})$ by

$$\bar{\chi}(x_0, y_1) := \left(x_0 + X(x_0, y_1)Bk, y_1 k + Y(x_0, y_1)\right).$$

Denote, from (1.6),

$$L^{-1} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+m}); \tag{2.5}$$

define $(\mathcal{F}_0 \times \mathcal{G}_0) := L^{-1}(\mathcal{F} \times \mathcal{G})$; and note from [2] that $U = \mathrm{R}(A_1)$. Using Theorem 7 from [2] we find that

$$C_1 Bk = k, \mathrm{N}(A_1) = \langle Bk \rangle, \quad Bk \notin \mathrm{R}(A_1).$$

Recall also that the restricted map $A_0 := A_1|_{\mathrm{R}(A_1)} \in GL(U)$ satisfies $\sigma(M, L) = \sigma(A_0^{-1})$.

Multiplying (2.4) by L^{-1} we can write (1.1)–(1.2) as

$$A_1 \dot{x} = x + \mathcal{F}_0(x, y), \tag{2.6}$$

$$C_1 \dot{x} = y + \mathcal{G}_0(x, y). \tag{2.7}$$

By forming the decomposition $\mathcal{F}_0(x, y) = \mathcal{F}_r(x, y) + \mathcal{F}_b(x, y)Bk \in U \oplus \langle Bk \rangle$, where $\mathcal{F}_b(x, y) = u^T C \mathcal{F}_0(x, y)/u^T CBk$ and $\mathcal{F}_r = \mathcal{F}_0 - \mathcal{F}_b Bk$, we obtain

$$\begin{aligned} A_1 \dot{x} &= A_1(\dot{x}_1 Bk + \dot{x}_0) \\ &= A_1 \dot{x}_0 \\ &= x_1 Bk + x_0 + \mathcal{F}_b(x, y)Bk + \mathcal{F}_r(x, y). \end{aligned}$$

By projecting this onto U along $\langle Bk \rangle$, we then obtain

$$A_1 \dot{x}_0 = x_0 + \mathcal{F}_r(x_0 + X(x_0, y_1)Bk, Y(x_0, y_1) + y_1 k).$$

But A_0 is the restriction of A_1 to $\mathrm{R}(A_1)$, so that

$$\dot{x}_0 = A_0^{-1} x_0 + \rho(x_0, y_1), \tag{2.8}$$

where $\rho(x_0, y_1) = A_0^{-1} \mathcal{F}_r(x_0 + X(x_0, y_1)Bk, Y(x_0, y_1) + y_1 k)$ is a C^ω function and $\mathcal{O}(2)$ at the origin.

From (2.7) one may write $C_1 \dot{x} = \dot{x}_1 C_1 Bk + C_1 \dot{x}_0 = y_0 + y_1 k + \mathcal{G}_0(x, y)$. Taking the inner product of this with k yields

$$\dot{x}_1 + k^T C_1 \dot{x}_0 = y_1 + k^T \mathcal{G}_0(x, y),$$

recalling that $k^T k = 1$. This implies

$$\dot{x}_1 + k^T C_1 [A_0^{-1} x_0 + \rho(x_0, y_1)] = y_1 + \kappa(x_0, y_1),$$

where $\kappa(x_0, y_1) = k^T \mathcal{G}_0(x_0 + X(x_0, y_1)Bk, Y(x_0, y_1) + y_1 k)$.

Now we find another expression for $\dot{x}_1$, using the fact that $y_1(\cdot) = k^T y(\cdot) \in W^{1,\infty}$ by assumption gives

$$\dot{x}_1 = \frac{d}{dt} X(x_0, y_1) = d_{x_0} X[\dot{x}_0] + d_{y_1} X[\dot{y}_1] = d_{x_0} X[A_0^{-1} x_0 + \rho(x_0, y_1)] + d_{y_1} X[\dot{y}_1];$$

then

$$\begin{aligned} & d_{x_0} X[A_0^{-1} x_0 + \rho(x_0, y_1)] + \dot{y}_1 d_{y_1} X[1] \\ & \qquad + k^T C_1 [A_0^{-1} x_0 + \rho(x_0, y_1)] = y_1 + \kappa(x_0, y_1). \end{aligned} \tag{2.9}$$

The proof is essentially complete, but to simplify the notation a little, let us write

$$L_0 := A_0^{-1}, \quad p := x_0, \quad q := y_1, \quad \bar{s}(p,q) := d_q X(p,q)[1],$$

and $a := -k^T C_1 A_0^{-1}$. From (2.9) we find a function r, given by $r(p,q) = \kappa(p,q) - (k^T C_1 + d_{x_0} X(p,q))\rho(p,q) - d_{x_0} X(p,q) A_0^{-1} p$, such that

$$\dot{p} = L_0 p + \rho(p,q), \qquad \bar{s}(p,q)\dot{q} = a^T p + q + r(p,q). \tag{2.10}$$

We claim that

$$\bar{s}(0,0) = 0, \qquad d_q \bar{s}(0,0)[1] = -u^T d_{yy}^2 g(0,0)[k,k]/u^T CBk, \tag{2.11}$$

and

$$d_p \bar{s}(0,0)[p] = -u^T d_{xy}^2 g(0,0)[p,k]/u^T CBk \tag{2.12}$$

for all $p \in U$ and $q \in \mathbb{R}$. To prove this claim, we use the fact that

$$g(X(p,q)Bk + p, qk + Y(p,q)) \equiv 0;$$

differentiating and evaluating this expression at zero yield (2.11) and (2.12).

Now define new coordinates $(\alpha, \beta) := (p, a^T p + q)$, and let

$$\chi(\alpha,\beta) := \bar{\chi}(\alpha, \beta - a^T\alpha), \qquad s(\alpha,\beta) := \bar{s}(\alpha, \beta - a^T\alpha).$$

This provides the C^ω functions ρ_0 and ρ_1 such that (p,q) satisfies (2.10) if and only if (α,β) satisfies (2.2)–(2.3).

Since $\mathbf{S}$ and $\chi(\Sigma)$ have dimension equal to $n-1$, to prove $\chi(\Sigma \cap B(0,0)) = \mathbf{S}$ it suffices to prove that $\chi(\Sigma \cap B(0,0)) \subset \mathbf{S}$, and we know from Lemma 2.1 that $\mathbf{C}$ is an n-dimensional manifold containing $(0,0)$ and $\mathbf{S}$ is a codimension-1 submanifold of $\mathbf{C}$, also containing $(0,0)$. Thus, let $(x,y) = (x_0 + x_1 Bk, y_0 + y_1 k) \in \mathbf{C}\backslash\mathbf{S}$ satisfy $s(x_0, y_1) = 0$. One can solve $g(x,y) = 0$ uniquely for $y = y(x)$ near this point by the implicit function theorem. Hence, locally, $g(x,y_1) = g(x,y_2) = 0$ implies $y_1 = y_2 = y(x)$.

Define the smooth function $w : \mathbb{R}^2 \to \mathbb{R}$ by

$$w(\theta,\tau) := \theta - X(x_0,\tau),$$

and note that $w(x_1,y_1) = 0$. By definition, $d_\tau w(x_1,y_1) = -s(x_0,y_1) = 0, d_\theta w(x_1,y_1) = 1$, and when (x,y) is of sufficiently small norm we may assume without the loss of any generality that $d_{\tau\tau}^2 w(x_1,y_1) \neq 0$ because $d_q s(0,0) \neq 0$. By the saddle-node bifurcation theorem there are two distinct solution branches of $w(\theta,\tau) = 0$ on which $\tau = \tau_\pm(\theta)$, say. Now suppose that a sequence $(x_1^m) \subset \mathbb{R}$ satisfies $x_1^m \to x_1$ as $m \to \infty$, so that the two sequences in $\mathbb{R}^{n+m}$ given by $((x_0 + \tau_\pm(x_1^m)Bk, y_0 + y_1 k))_m$ lie in $\mathbf{C}\backslash\mathbf{S}$ for m large enough. By uniqueness it follows that $\tau_+(x_1^m) \equiv \tau_-(x_1^m)$, a contradiction. Therefore, no such (x,y) exists and the result is proven. ∎

In light of Theorem 2.2, we define the following terminology. Suppose that $I \subset \mathbb{R}$ is a bounded, open interval. We call a map $(\alpha,\beta) \in C^1(I,U) \times W^{1,\infty}(I,\mathbb{R})$ a *sharp solution* of (2.2)–(2.3) if this differential equation holds for almost every $t \in I$, provided that $\dot{\beta} \notin C^0(I,\mathbb{R})$. A map $(\alpha,\beta) \in C^1(I, U \times \mathbb{R})$ is said to be a *smooth solution* of (2.2)–(2.3) if this differential equation is satisfied for all $t \in I$.

We assume throughout, without the loss of any generality, that $d_\beta s(\alpha,\beta) \neq 0$ for all $(\alpha,\beta) \in \Sigma$. Due to the fact that Σ is diffeomorphic to $\mathbf{S}$ and because existence and uniqueness of (2.2)–(2.3) may break down along Σ, we shall also describe Σ as *the singularity.*

For the moment let us record the fact, taken from the above proof, that

$$d_\alpha s(0,0)[p] = -u^T \left(d^2_{xy} g(0,0)[p,k] - a^T p \; d^2_{yy} g(0,0)[k,k]\right)/u^T CBk,$$

where k is defined in A2 and a is given in the proof. Let us note that the following assumption ensures that $d_\alpha s(0,0)$ is a nonzero map:

A6. $\exists p' \in U$ such that $d^2_{xy} g(0,0)[p',k] - a^T p' \; d^2_{yy} g(0,0)[k,k] \notin \mathrm{R}(d_y g(0,0))$.

Using Theorem 2.1 of [9] we can describe the impasse points of (2.2)–(2.3) as follows.

LEMMA 2.3. *Assuming* A1–A6*, if* $(\alpha,\beta) \in \Sigma$ *satisfies* $\beta + \rho_1(\alpha,\beta) \neq 0$*, then* (α,β) *is an impasse point for* (2.2)–(2.3).

Therefore, the set

$$P := \{(\alpha,\beta) \in \Sigma : \beta + \rho_1(\alpha,\beta) = 0\} \tag{2.13}$$

forms a subset of the singularity which does not necessarily contain impasse points, but the following lemma shows that P represents a nongeneric set of singular points.

LEMMA 2.4. *Assuming* A1–A6*, the set of pseudoequilibria of* (2.2)–(2.3), $P \subset B(0,0)$*, is a codimension-*1 *submanifold of* Σ.

Proof. Use the implicit function theorem to solve the system $\beta + \rho_1(\alpha,\beta) = 0$, $s(\alpha,\beta) = 0$ near $(\alpha,\beta) = (0,0)$. ∎

Nevertheless, the following result shows that (2.2)–(2.3) is well behaved at P in the sense that there exists a smooth solution of this quasilinear ODE through every point in P.

THEOREM 2.5. *Suppose that* A1–A6 *hold and let* $r \in \mathbb{N}$. *There is a neighborhood* $B^{(r)}(0,0) \subset B(0,0)$ *and at least one* $(n-1)$*-dimensional, quasi-invariant* C^r *manifold* $W^R \subset B^{(r)}(0,0)$ *of* (2.2)–(2.3) *such that for each* $(\alpha_0,\beta_0) \in W^R$*, there exists an open interval* $I \ni 0$ *and a unique* C^r*-solution of* (2.2)–(2.3), $(\alpha,\beta) : I \to W^R$ *such that* $(\alpha(0),\beta(0)) = (\alpha_0,\beta_0)$. *Moreover,* $W^R \cap \Sigma = P$.

Proof. Make the following change of time-scale: if $(\alpha(t),\beta(t))$ satisfies (2.2)–(2.3), define τ by

$$\frac{d\tau}{dt} = \frac{1}{s(\alpha(t),\beta(t))}, \qquad \tau(t_0) = \tau_0,$$

and write $\alpha(\tau) = \alpha(t(\tau)), \beta(\tau) = \beta(t(\tau))$. If a prime denotes $\frac{d}{d\tau}$, then

$$\alpha' = (L_0\alpha + \rho_0(\alpha,\beta))s(\alpha,\beta), \tag{2.14}$$

$$\beta' = \beta + \rho_1(\alpha,\beta). \tag{2.15}$$

Linearizing (2.14)–(2.15) around the equilibrium point $(\alpha,\beta) = (0,0)$, we find *at least one* C^r, local center manifold $W^R := W^c_{loc}$. This is a quasi-invariant manifold for (2.2)–(2.3) on which $\beta = h(\alpha)$, where $h(0) = 0$ and $dh(0) = 0$. Now suppose that $s(\alpha_0,\beta_0) = 0$ and $(\alpha_0,\beta_0) \in W^R$, and let $(\alpha(\tau),\beta(\tau))$ be the solution of (2.14)–(2.15) in W^R with $(\alpha(0),\beta(0)) = (\alpha_0,\beta_0)$. Then

$$\beta + \rho_1(\alpha,\beta) = \beta' = dh(\alpha)\alpha' = dh(\alpha)(L_0\alpha + \rho_0(\alpha,\beta))s(\alpha,\beta),$$

and setting $\tau = 0$ shows that $W^R \cap \Sigma \subseteq P$. However, the left-hand side of this inclusion is given by those α for which $s(\alpha, h(\alpha)) = 0$. This equation can be solved by the implicit function theorem, showing that $W^R \cap \Sigma$ is also an $(n-2)$-dimensional manifold. Since $W^R \cap \Sigma$ and P are manifolds of the same dimension and one is contained in the other, they coincide. The uniqueness of solutions of (2.2)–(2.3) in W^R follows from a standard ODE uniqueness theorem applied to $\dot{\alpha} = L_0\alpha + \rho_1(\alpha, h(\alpha))$, with $\beta(t) = h(\alpha(t))$. ∎

While the existence of W^R is assured from the center manifold theorem, it is not clear that there will be only one W^R with the properties outlined in Theorem 2.5. For this reason, we cannot claim that W^R is an invariant manifold, we can claim only quasi-invariance.

The following definition is given merely for completeness, and it provides the analogy of stable and unstable manifolds for (2.2)–(2.3).

DEFINITION 1 (local stable and unstable sets). *Let $B' \subset U \times \mathbb{R}$ be a neighborhood of $(0,0)$. The local stable set $W^s(0,0) \subset U \times \mathbb{R}$ is the set of $(\alpha, \beta) \in B'$ such that there exists a solution $(\alpha(t), \beta(t))$ of (2.2)–(2.3) with $(\alpha(0), \beta(0)) = (\alpha, \beta)$, $(\alpha(t), \beta(t)) \in B'$ for all $t \geq 0$ and $(\alpha(t), \beta(t)) \to 0$ as $t \to \infty$. The local unstable set $W^u(0,0)$ is defined analogously with $t \leq 0$ and the limit $t \to -\infty$ used above.*

PROPOSITION 2.6. *Suppose that A1–A6 hold and that (M, L) is a hyperbolic matrix pencil. Now define*

$$n_{\pm} := \#\left(\sigma(M, L) \cap \mathbb{C}^{\pm}\right),$$

both assumed to be nonzero, noting $n_- + n_+ = n-1$. Then there is an invariant subset of the stable set of (2.2)–(2.3), $W^{Rs} \subset W^R$, which is an (n_-)-dimensional manifold, and an invariant subset of the unstable set of (2.2)–(2.3), $W^{Ru} \subset W^R$, which is an (n_+)-dimensional manifold.

Proof. This uses the existence of the quasi-invariant manifold, W^R, of (2.2)–(2.3) on which $\beta = h(\alpha)$. The result follows since the ODE $\dot{\alpha} = L_0\alpha + \rho_1(\alpha, h(\alpha))$ has stable and unstable manifolds of the stated dimensions and using the fact that $\sigma(M, L) = \sigma(L_0)^{-1}$ from Theorem 2.2. ∎

Let us note that the fact that the stable and unstable sets $W^{s,u}(0,0)$ associated with (2.2)–(2.3) are not necessarily manifolds is simply due to the ellipticity of the zero equilibrium of (2.14)–(2.15).

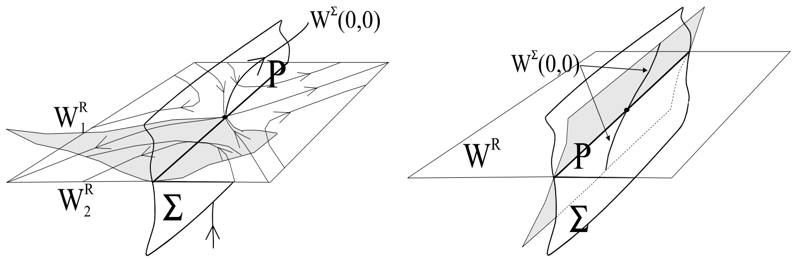
Now we use the remaining information in the normal form (2.2)–(2.3) to deduce that not only are there singularity-traversing solutions contained in W^R, there are other quasi-invariant manifolds which intersect the singularity Σ.

PROPOSITION 2.7. *Suppose that A1–A6 apply. Associated with each $(\alpha, \beta) \in P$ is a C^ω, one-dimensional, quasi-invariant manifold of (2.2)–(2.3), $W^\Sigma(\alpha, \beta)$, which is transverse to both W^R and Σ at (α, β). Moreover, if $(\alpha_0, \beta_0) \in W^\Sigma(0,0) \backslash (0,0)$, there exists a $T \in \mathbb{R}$ and a solution $(\alpha(t), \beta(t))$ of (2.2)–(2.3) on $[0, T]$ such that $(\alpha(0), \beta(0)) = (\alpha_0, \beta_0)$ and $\operatorname{sign}\ s(\alpha(T), \beta(T)) = -\operatorname{sign}\ s(\alpha(0), \beta(0))$.*

Proof. Suppose that $(\alpha, \beta) \in P$, so that (α, β) is an equilibrium of (2.14)–(2.15). Linearizing (2.14)–(2.15) around this equilibrium gives a smoothly parameterized mapping $T \in C^\omega\left(B(0,0), \mathcal{L}\left(U \times \mathbb{R}\right)\right)$ such that

$$T(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $B(0,0)$ is defined in Theorem 2.2. Since 1 is an algebraically simple eigenvalue of $T(0,0)$, by spectral perturbation results [5] there are C^ω functions $\lambda : B(0,0) \to \mathbb{R}$



(a) (b)

FIG. 2.1. *A typical flow near a singular equilibrium (solid dot).* (a) *Two instances of* W^R, $W^R_{1,2}$ *are shown, where* $W^\Sigma(0,0)$ *and* Σ *are shown transverse at* $(0,0)$. (b) *The relative positions of* Σ, W^R, *and* $W^\Sigma(0,0)$*; the shaded set is* $\bigcup_{(\alpha,\beta)} W^\Sigma(\alpha,\beta)$. *Elements of* $\Sigma\backslash P$ *are impasse points.*

and $e : B(0,0) \to U \times \mathbb{R}$ such that $\lambda(\alpha,\beta) \in \sigma(T(\alpha,\beta))$, with corresponding unit eigenvector $e(\alpha,\beta)$, such that $e(0,0) = (0,1)$ and $\lambda(0,0) = 1$. Hence we may assume with loss of generality that $\lambda(\alpha,\beta)$ is positive whenever $(\alpha,\beta) \in P \cap B(0,0)$. From this it follows that each $(\alpha,\beta) \in P$ has an associated local unstable manifold, $W^u(\alpha,\beta)$, which we write as $W^\Sigma(\alpha,\beta)$.

The representation of $W^\Sigma(0,0)$ is given by a graph of the form $\alpha = \ell(\beta)$, such that $\ell(0) = 0$ and $d\ell(0) = 0$. Therefore, the solutions of (2.2)–(2.3) on $W^\Sigma(0,0)$ are images of the solutions of the scalar ODE

$$\dot{\beta} = \frac{\beta + \rho_1(\ell(\beta),\beta)}{s(\ell(\beta),\beta)} = d_\beta s(0,0)^{-1} + O(\beta),$$

and the right-hand side of this is nonzero in a neighborhood of $\beta = 0$. Hence the solution passes through the regular point $\beta = 0$ in finite time.

Since $e(\cdot,\cdot)$ varies smoothly, it follows without the loss of any generality that each $W^\Sigma(\alpha,\beta)$ is transverse to Σ if $W^\Sigma(0,0)$ is transverse to Σ. Therefore, let us calculate $T_0(\Sigma)$, given that $T_0(W^\Sigma(0,0)) = U \times \{0\} \subset U \times \mathbb{R}$. Since we may solve $s(\alpha,\beta) = 0$ near $(0,0)$ for $\beta = \beta(\alpha)$ such that $s(\alpha,\beta(\alpha)) \equiv 0$, we find $d\beta(0) = -d_\beta s(0,0)^{-1} d_\alpha s(0,0) \neq 0$ and $T_0(\Sigma) = \{(\alpha, d\beta(0)\alpha) : \alpha \in U)\}$. It follows that $\dim(T_0(\Sigma) \oplus T_0(W^\Sigma(0,0))) = n$ and therefore the manifolds Σ and $W^\Sigma(\alpha,\beta)$ intersect transversally at (α,β). ∎

In [11], the authors use W^{sing} to denote a one-dimensional, quasi-invariant manifold containing the singular equilibrium. We use $W^\Sigma(\alpha,\beta)$ (and $W^\Sigma(0,0)$ is W^{sing}) to underline the fact that through every point on this set, there is a solution which can be extended to the singularity Σ. (See Figure 2.1(b).)

Let us note that it is possible for a subset of $W^\Sigma(0,0)$ to lie in either the stable or unstable set associated with (2.2)–(2.3). Indeed, we shall define

$$W^{\Sigma s}(0,0) := W^\Sigma(0,0) \cap W^s(0,0),$$

and similarly $W^{\Sigma u}(0,0) := W^\Sigma(0,0) \cap W^u(0,0)$. Theorem 2.8 below shows that $W^{\Sigma s}$ and $W^{\Sigma u}$ are not empty if A1–A6 apply, because of the existence of sharp solutions.

THEOREM 2.8. *Assuming* A1–A6*, through the singular equilibrium of* (2.2)–(2.3) *there pass two smooth solutions with* $(\alpha,\beta) \in C^\omega \times C^\omega$ *and two sharp solutions with* $(\alpha,\beta) \in C^1 \times W^{1,\infty}$. *Consequently,* $W^{\Sigma u}(0,0)$ *and* $W^{\Sigma s}(0,0)$ *are nonempty.*

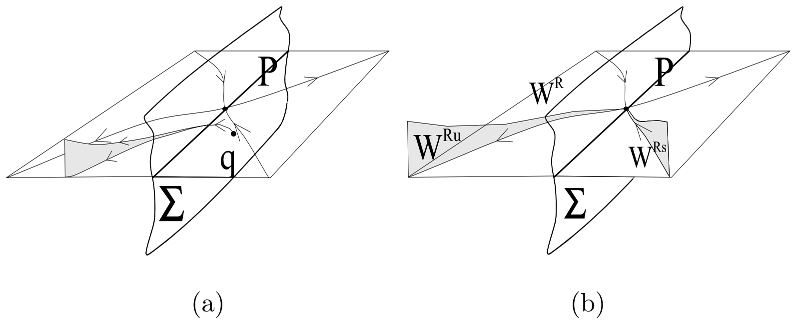


FIG. 2.2. *A typical flow near a singular equilibrium:* (a) *There are multiple instances of* W^R*, showing the possible lack of uniqueness of solutions along* $P = W^R \cap \Sigma$*. The shaded region shows the union of all possible forward orbits of* q *after encountering the singularity.* (b) *The local stable* (W^{Rs}) *and unstable* (W^{Ru}) *sets associated with* $(0,0)$.

Proof. One of the $C^\omega \times C^\omega$ solutions is the trivial equilibrium solution itself. The other smooth solution is obtained from the trajectory of (2.2)–(2.3) whose image forms $W^\Sigma(0,0)$.

Now concatenate a trajectory of (2.2)–(2.3) with initial condition on $W^\Sigma(0,0)$ to the equilibrium solution to form two sharp solutions. To see that this procedure forms a sharp solution, consider the solution (α,β) of (2.2)–(2.3) which is zero for $t \geq 0$ but lies on $W^\Sigma(0,0)$ for $t \in (-T,0]$. Then, for $t \leq 0$, from the proof of Proposition 2.7 we have $\alpha = \ell(\beta)$, where $\ell(0) = 0$ and $d\ell(0) = 0$, but $\dot\beta = (\beta + \rho_1(\ell(\beta),\beta))/s(\ell(\beta),\beta)$. It follows that $\dot\beta(0_-) = 1/d_\beta s(0,0) \neq 0$ but $\dot\beta(0_+) = 0$, and $\dot\beta \in L^\infty(-T,T)$ for small enough $T > 0$. Another sharp solution is obtained by starting on the equilibrium solution before leaving the equilibrium along $W^\Sigma(0,0)$ in an analogous manner. ∎

An illustration of the invariant manifolds discussed in this section is given in Figures 2.1 and 2.2.

3. Discussion. Let us consider two examples which illustrate some of the fundamental ideas within the paper. The first example is somewhat artificial, but it clearly shows how smooth solutions can be concatenated to form less regular ones.

Example 1. Consider

$$\dot{x} = y, \qquad x^2 + y^2 = 1, \tag{3.1}$$

where $(x,y) = (\pm 1, 0)$ are both singular equilibrium points, so that there exists a trivial smooth solution passing through them. However, $(x(t),y(t)) = (\cos(t),\sin(t))$ is another smooth solution passing through these two points; this is precisely the behavior we observe at a singular equilibrium in higher dimensions, with $W^\Sigma(0,0)$ playing the role of the circle of this example. The concatenated function

$$(x(t),y(t)) = \begin{cases} (1,0), & t \leq 0, \\ (\cos(t),\sin(t)), & t \geq 0, \end{cases}$$

is a solution of (3.1) of class $C^1 \times W^{1,\infty}$. Indeed, because $\dot y(0_-) = 0$ and $\dot y(0_+) = -1$ we have $y \notin C^1(\mathbb{R})$, although $\dot y \in L^\infty(\mathbb{R})$.

This example demonstrates that the multiplicity of sharp solutions can be much greater than that of smooth solutions. This arises in this particular instance because of the existence of a connecting orbit between the two singular equilibria. So, in order to form a continuum of sharp solutions, one can simply "wait" for some arbitrary time on arrival at the singular equilibrium before continuing around the circle to the other singular equilibrium.

Example 2. Consider a degenerate form of the *Fitzhugh–Nagumo* equation:

$$u_t = \frac{1}{2}(u^2)_{xx} + u(1-u) + v, \tag{3.2}$$

$$v_t = u - v, \qquad x \in \mathbb{R}, t > 0. \tag{3.3}$$

Scalar problems of this type can be found in [4, 10], where the authors are interested in the support of the waves, which may be finite, semi-infinite, or infinite. This equation is related to the much-studied Fitzhugh–Nagumo equation, except for the inclusion of the degenerate diffusive term. By seeking a traveling-wave solution of (3.2)–(3.3) which connects $(u, v) = (0, 0)$ to itself, we obtain the quasilinear problem

$$\left(cu - \frac{1}{2}(u^2)_z\right)_z = u(1-u) + v, \tag{3.4}$$

$$cv_z = u - v, \qquad u(\pm\infty), v(\pm\infty) = 0,\ z = x + ct, \tag{3.5}$$

where $c > 0$ is the wave speed. To study (3.4)–(3.5) as a DAE, we require $u \in C^0$ to also satisfy $u^2 \in C^1$ and $cu - \frac{1}{2}(u^2)_z \in C^1$, rather than simply allowing $u \in C^2$. This ensures that the resulting solutions are *weak* solutions if one considers (3.4)–(3.5) in a standard weak formulation [6]. A simple example of a function which satisfies such a regularity requirement is $u(t)$, where $u(t) = 0$ for $t < 0$ and $u(t) = t$ for $t \geq 0$ and $c = 1$.

We can manipulate this system to see explicitly how the results of the previous sections apply in this specific case. Thus, put $U = u - W$ and write (3.4)–(3.5) as a DAE,

$$w_z = cW, \tag{3.6}$$

$$cU_z = (W + U)(1 - W - U) + v, \tag{3.7}$$

$$cv_z = W + U - v, \tag{3.8}$$

$$0 = w - \frac{1}{2}(U + W)^2, \tag{3.9}$$

to which A1–A6 apply if $c \neq 0$. The constraint manifold for this problem is $\mathbf{C} = \{(w, U, v, W) \in \mathbb{R}^4 : (U + W)^2 = 2w\}$, and the singularity is $\mathbf{S} = \{(w, U, v, W) \in \mathbf{C} : U + W = 0\}$. Differentiating (3.9), we obtain a quasilinear ODE which is analogous to (2.2)–(2.3):

$$(U + W)W_z = cW - (U + W)[(U + W)(1 - U - W) - v]c^{-1}, \tag{3.10}$$

$$cv_z = W + U - v, \tag{3.11}$$

$$cU_z = (W + U)(1 - W - U) + v, \tag{3.12}$$

which, upon rescaling time, gives

$$W' = cW - (U + W)[(U + W)(1 - U - W) - v]c^{-1}, \tag{3.13}$$

$$cv' = (U + W)[W + U - v], \tag{3.14}$$

$$cU' = (U + W)((W + U)(1 - W - U) + v). \tag{3.15}$$

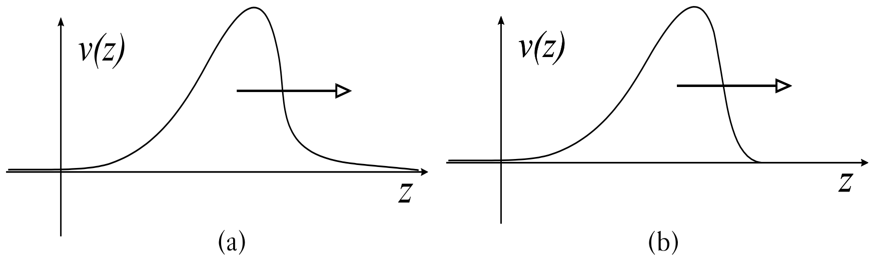


FIG. 3.1. (a) *A solution with a head and a tail when* W^{Rs} *intersects* W^{Ru}. (b) *A solution with a tail but no head when* $W^{\Sigma s}$ *intersects* W^{Ru}. *In this case, the solution is identically zero ahead of the wave.*

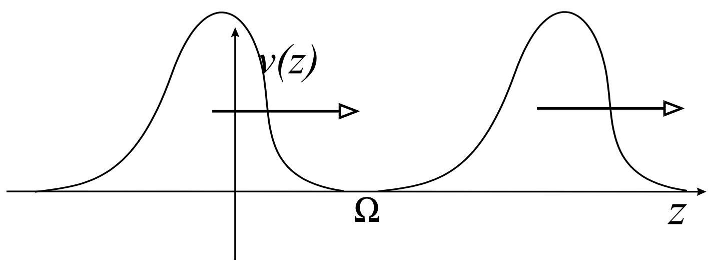


FIG. 3.2. *A solution when* $W^{\Sigma s}$ *intersects* $W^{\Sigma u}$, *and multiple waves arise. The solution is zero between each wave in the region* Ω.

It is straightforward to show that at the singular equilibrium point $(W, v, U) = (0, 0, 0)$, (3.10)–(3.12) has at least one quasi-invariant manifold, W^R, described by a graph of the form $W = h(U, v)$. For $c > 0$, there is another quasi-invariant manifold, $W^{\Sigma}(0, 0)$, on which $U = H_1(W)$ and $v = H_2(W)$. Now, restricting (3.4)–(3.5) induces a local dynamical system on W^R, given by the restricted flow of an ODE of the form

$$cv_z = U - v + \mathcal{O}(2), \tag{3.16}$$

$$cU_z = U + v + \mathcal{O}(2). \tag{3.17}$$

Since the equilibrium of this system, $(U, v) = (0, 0)$, has a stable and unstable manifold, it follows that (3.4)–(3.5) has at least two one-dimensional invariant manifolds, W^{Rs} and W^{Ru}, within its stable and unstable sets. The arrival time of solutions at the zero equilibrium along these manifolds must be infinite, by standard ODE uniqueness results, as applied to (3.16)–(3.17).

On $W^{\Sigma}(0, 0)$, using (3.10), we have an ODE

$$W_z = \frac{cW + O(W^2)}{W + H_1(W)} = c + O(W)$$

for small $|W|$. One can verify directly that $\mathbf{S}$ and $W^{\Sigma}(0, 0)$ intersect transversally in $\mathbf{C}$, so that there is at least one singularity-traversing smooth solution of (3.4)–(3.5). It follows that there are also sharp solutions which start and end at the equilibrium, existing on either side of the singularity. These are again formed by concatenating the trivial equilibrium solution to the trajectory which forms $W^{\Sigma}(0, 0)$.

While this does not provide any information as to whether a homoclinic orbit exists in (3.4)–(3.5), the intersections of the various manifolds involved will yield different types of traveling waves, as depicted in Figures 3.1 and 3.2.

Acknowledgments. Many thanks are due to the anonymous referees whose comments helped to improve an earlier version of the paper.

REFERENCES

[1] R. E. Beardmore, *Stability and bifurcation properties of index-*1 *DAEs*, Numer. Algorithms, 19 (1998), pp. 43–53.

[2] R. E. Beardmore, *The singularity-induced bifurcation and its Kronecker normal form*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 126–137.

[3] S. Campbell, R. Hollenbeck, and W. Marszalek, *Mixed symbolic-numeric computations with general DAE* I: *System properties*, Numer. Algorithms, 19 (1998), pp. 73–84.

[4] A. de Pablo and A. Sanchez, *Global travelling waves in reaction-convection diffusion equations*, J. Differential Equations, 165 (2000), pp. 377–413.

[5] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer-Verlag, New York, 1980.

[6] M. Langlais and D. Phillips, *Stabilization of solutions of nonlinear and degenerate evolution equations*, Nonlinear Anal., 9 (1985), pp. 321–333.

[7] R. März, *Criteria for the trivial solution of differential algebraic equations with small nonlinearities to be asymptotically stable*, J. Math. Anal. Appl., 225 (1998), pp. 587–607.

[8] P. J. Rabier and W. C. Rheinboldt, *A geometric treatment of quasilinear differential-algebraic equations*, J. Differential Equations, 109 (1994), pp. 110–146.

[9] P. J. Rabier and W. C. Rheinboldt, *On impasse points of quasilinear differential-algebraic equations*, J. Math. Anal. Appl., 181 (1994), pp. 429–454.

[10] F. Sanchez-Garduno and P. K. Maini, *Travelling-wave phenomena in some degenerate reaction-diffusion equations*, J. Differential Equations, 117 (1995), pp. 281–319.

[11] V. Venkatasubramanian, H. Schattler, and J. Zaborszky, *A Stability Theory of Large Differential Algebraic Systems, a Taxonomy*, Tech. report SSM 9201, Part 1, Washington University, St. Louis, MO, 1992.

[12] H. von Sosen, *Part* I: *Folds and Bifurcations in the Solutions of Semi-Explicit DAEs*, Ph.D. dissertation, California Institute of Technology, Pasadena, CA, 1994.