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THE SINGULARITY-INDUCED BIFURCATION AND ITS KRONECKER NORMAL FORM*

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Abstract. It is shown that the singularity-induced bifurcation theorem due to Venkatasubramanian, Schattler, and Zaborszky [*Proceedings of the IEEE*, 83 (1995), pp. 1530–1558] can be expressed as the perturbation of an infinite eigenvalue of a particular class of parameterized index-1 matrix pencil, denoted $(M, L(\lambda))$. It is shown that the matrix pencil at the singularity-induced bifurcation point, $(M, L(\lambda_0))$, has Kronecker index-2. It is also shown that a two-parameter unfolding of a singularity-induced bifurcation point results in a locus of index-0 pencils, denoted $(M(\epsilon), L(\lambda(\epsilon)))$, which has two purely imaginary eigenvalues near infinity.

Key words. unfoldings, bifurcation, matrix pencil

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1. Introduction.

Consider the parameterized semiexplicit, index-1 differential algebraic equation (DAE)

$$\dot{x} = f(x, y, \lambda), \tag{1.1a}$$

$$0 = g(x, y, \lambda), \tag{1.1b}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, and $\lambda \in \mathbb{R}$. As the parameter λ varies, an equilibrium of (1.1) may encounter the *singular set*

$$S = \{(x, y) \in \mathbb{R}^{n+m} : g(x, y, \lambda) = 0,\ \det(d_y g(x, y, \lambda)) = 0\}.$$

The singularity-induced bifurcation theorem (SIB), which was first proven in [16, 17], describes the behavior of an eigenvalue associated with the linearization of (1.1) at such an encounter.

For completeness, we state the SIB theorem from [17].

THEOREM 1 (SIB). *Define*

$$\Delta(x, y, \lambda) = \det(d_y g(x, y, \lambda)),$$

set $z = (x, y)$ *and let* $F(x, y, \lambda) = (f \times g)(x, y, \lambda)$. *Suppose that* z_0 *is an equilibrium of* (1.1) *and write* $w = (z_0, \lambda_0)$. *If*

1. $d_y g(w)$ *has a simple zero eigenvalue,*
2. *the matrix* $\begin{pmatrix} d_z F(w) & d_\lambda F(w) \\ d_z \Delta(w) & d_\lambda \Delta(w) \end{pmatrix}$ *is invertible,*
3. $tr(d_y f(w) \cdot \mathrm{adj}(d_y g(w)) \cdot d_x g(w)) \neq 0$, *and*
4. $d_z F(w)$ *is invertible,*

then there is a unique, parameterized equilibrium of (1.1), *defined in a neighborhood of* λ_0, *which is transversal to the singular set* S *at* λ_0. *One real eigenvalue of the*

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linearization of (1.1) *about this equilibrium locus moves from one open half of the complex plane to the other by diverging to infinity as* λ *passes* λ_0. *The remaining* $n-1$ *eigenvalues of the linearization lie in some bounded set which does not contain the origin.*

There are several ingredients in the method of proof of Theorem 1 used in [16] which are essentially nonlinear in nature. The approach used therein is to rescale time and subsequently rewrite (1.1) as a smooth, parameterized vector field. Venkatasubramanian, Schattler, and Zaborszky then apply the center manifold theorem to probe this vector field and the eigenvalues of its linearization in a neighborhood of the equilibrium which lies on S.

The described approach leads naturally to a description of the flow of the DAE (1.1) when the equilibrium lies on S, namely, when $\lambda = \lambda_0$. In [16] this description is called the local decomposition theorem (LDT). We shall show in a future paper [4] that the algebraic approach used in this paper, coupled to the implicit function theorem, can be used to derive an extension of the LDT. This is important due to the fact that the LDT is starting to find applications in wider mathematics [7, 14].

However, the SIB theorem concerns the perturbation of an infinite eigenvalue of a parameterized matrix pencil. This observation permitted a result equivalent to the SIB to be proven in [3]. This provided a much simpler method of proof and allowed hypothesis 4 of the SIB theorem to be removed, at the same time maintaining the asymptotic nature of the diverging eigenvalue.

We shall denote the diverging eigenvalue by $\alpha(\lambda)$ and in [16, 3] it was shown that there is some $\mu \neq 0$ such that

$$\alpha(\lambda) \sim \frac{\mu}{\lambda - \lambda_0} + O(1)$$

as $\lambda \to \lambda_0$. The point λ_0 is said to be a *singularity-induced bifurcation point.*

Let us define some notation. For any vector $v \in \mathbb{R}^p$ we shall write $\langle v \rangle = \{sv : s \in \mathbb{R}\}$. If $L : \mathbb{R}^p \to \mathbb{R}^p$ is a linear mapping, then $\sigma(L)$ denotes its spectrum, $\mathbf{N}(L)$ and $\mathbf{R}(L)$ denote its null-space and range, respectively. We shall use $\mathcal{N}(L)$ and $\mathcal{R}(L)$ to denote its generalized null-space and range, respectively. We shall use the symbol # to denote cardinality and $\mathcal{L}(\mathbb{R}^p)$ denotes the space of linear mappings over $\mathbb{R}^p$. We shall use Λ to denote some open interval containing λ_0.

Suppose that (R, S) is a matrix pencil over $\mathbb{R}^N$. It is said to be regular if there is some $s \in \mathbb{C}$ such that

$$\det(sR - S) \neq 0.$$

Moreover, we shall write

$$\sigma(R, S) = \{s \in \mathbb{C} : \det(sR - S) = 0\}.$$

The following is well known [9, 10, 6] and is the Kronecker normal form (KNF).

THEOREM 2 (KNF). *Suppose that* (R, S) *is a regular matrix pencil on* $\mathbb{R}^N$. *One can write* $\mathbb{R}^N = U \oplus V$ *and find nonsingular maps* $P \in \mathcal{L}(\mathbb{R}^N)$ *and* $Q \in \mathcal{L}(U \oplus V, \mathbb{R}^N)$ *such that*

$$PRQ = \begin{bmatrix} I_u & 0 \\ 0 & N \end{bmatrix} \text{ and } PSQ = \begin{bmatrix} C & 0 \\ 0 & I_v \end{bmatrix},$$

where there is an integer $k \geq 1$ *such that* $N^k = 0$ *and* $N^{k-1} \neq 0$.

Now $C, I_u \in \mathcal{L}(U)$, $N, I_v \in \mathcal{L}(V)$ *and* I_u *and* I_v *are identities on* U *and* V, *respectively. Moreover,* $\sigma(R,S) = \sigma(C)$ *and* $\#\sigma(R,S) = \dim U$. *Define the* index *of* (R,S), *written* $\mathrm{ind}(R,S)$, *to be the integer* k.

Suppose that (1.1) has a zero equilibrium for all $\lambda \in \Lambda$ and suppose that

$$L(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+m})$$

is the linearization of $f \times g$ about that equilibrium. In addition, we suppose that $\lambda \mapsto L(\lambda)$ is a C^r mapping. If $D(\lambda) = d_y g(0,0,\lambda)$ is invertible, the implicit function theorem implies that the eigenvalues of

$$\mathcal{S}_\lambda \stackrel{\text{def}}{=} A(\lambda) - B(\lambda)D^{-1}(\lambda)C(\lambda) \tag{1.2}$$

determine the linear stability properties of the trivial equilibrium of (1.1).

Define the matrix

$$M(\epsilon) \stackrel{\text{def}}{=} \begin{bmatrix} I_n & 0 \\ 0 & \epsilon I_m \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+m}),$$

where the n and m may be omitted for brevity. Write $M \stackrel{\text{def}}{=} M(0)$.

The paper is organized as follows. The purpose of section 2 is twofold. First we present a result which is equivalent to the SIB theorem, but with a method of proof that is different from both [16] and [3]. Moreover, the method of proof used in section 2 shows that the linearization

$$(M, L(\lambda))$$

of (1.1) has the following property. If $\lambda \neq \lambda_0$, then $(M, L(\lambda))$ has index-1. We compute the Kronecker normal form and show that $(M, L(\lambda_0))$ has index-2.

The following simple theorem is called the singular Hopf bifurcation (SHB).

THEOREM 3 (SHB). *Let* $a, b, c, d \in C^1(\mathbb{R})$ *and suppose that for some* $\lambda_0 \in \mathbb{R}$, $d(\lambda_0) = 0, d'(\lambda_0) \neq 0$, *and* $c(\lambda_0)b(\lambda_0) = -\omega_0^2 < 0$. *Then there is an interval* $I = (0, \epsilon_0) \subset \mathbb{R}$ *and* C^1 *functions* $\omega, \lambda : I \to \mathbb{R}$ *such that for all* $\epsilon \in I$

$$i\omega(\epsilon) \in \sigma \begin{pmatrix} a(\lambda(\epsilon)) & b(\lambda(\epsilon)) \\ \epsilon^{-1}c(\lambda(\epsilon)) & \epsilon^{-1}d(\lambda(\epsilon)) \end{pmatrix}.$$

Moreover, $\lambda(0_+) = \lambda_0$ *and* $\lim_{\epsilon \to 0_+} \epsilon^{1/2}\omega(\epsilon) = \omega_0$.

The SHB theorem plays a central role in [2, 15] and relates to the many discussions of duck solutions and ducks oscillations to be found, for instance, in [5, 18, 8, 1]. Motivated by Theorem 3, in section 3 we consider a singular unfolding of the linearization of (1.1), given by

$$M(\epsilon)\dot{z} = L(\lambda)z. \tag{1.3}$$

It is shown that the infinite eigenvalue from the SIB theorem perturbs to give a pair of purely imaginary eigenvalues of $(M(\epsilon), L(\lambda))$ near infinity.

2. KNF and the SIB. We now present several preliminary lemmas.

LEMMA 1. *Let* $\det : \mathcal{L}(\mathbb{R}^q) \to \mathbb{R}$ *be the determinant functional. It has derivative* $\det'$ *given by* $\det'(P)[H] = \mathrm{tr}(\mathrm{adj}P \cdot H)$, *where* $P, H \in \mathcal{L}(\mathbb{R}^q)$.

LEMMA 2. *If* $\det D(\lambda) \neq 0$, *then* $\dim \mathbf{N}(\mathcal{S}_\lambda) = \dim \mathbf{N}(L(\lambda))$ *and* $\det \mathcal{S}_\lambda \cdot \det D(\lambda) = \det L(\lambda)$.

LEMMA 3. *Let* $P \in \mathcal{L}(\mathbb{R}^q)$ *and suppose that there is a nonzero* $k \in \mathbb{R}^q$ *such that* $\mathbf{N}(P) = \langle k \rangle$, *then* $\mathbf{R}(\mathrm{adj}P) = \langle k \rangle$ *and* $\mathbf{N}(\mathrm{adj}P) = \mathbf{R}(P)$.

We shall often omit the reference to λ_0 for brevity and simply write

$$L(\lambda_0) = L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+m}).$$

LEMMA 4. *Suppose* $L \in \mathcal{L}(\mathbb{R}^{n+m})$ *is invertible and has its inverse decomposed as*

$$L^{-1} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+m}).$$

Then the finite spectrum of the pencil (M, L) *satisfies* $\sigma(M, L) = [\sigma(A_1) \backslash \{0\}]^{-1}$.

Proof. Clearly

$$\begin{aligned} \det(\lambda M - L) = 0 &\Leftrightarrow \det(L^{-1}M - \lambda^{-1} I) = 0 \\ &\Leftrightarrow \det \begin{bmatrix} A_1 - \lambda^{-1} I_n & 0 \\ C_1 & -\lambda^{-1} I_m \end{bmatrix} = 0 \end{aligned}$$

and the result follows since $\det L \neq 0$. □

We shall also use $A_1(\lambda), B_1(\lambda)$, and so on, to denote the elements in the decomposition of the parameterized inverse $L(\lambda)^{-1}$.

LEMMA 5. *If* $\det D(\lambda) \neq 0$, *then* $\sigma(\mathcal{S}_\lambda) = \sigma(M, L(\lambda))$.

Lemma 4 shows that we can understand the infinite eigenvalue of (M, L) by studying the zero eigenvalues of the mapping A_1. Before studying the implications of Lemma 4 further, we present the generalized SIB theorem [3].

THEOREM 4. *Suppose that* $r \geq 3$ *and* $L \in C^r(\Lambda, L(\mathbb{R}^{n+m}))$. *Suppose further that* $\mathbf{N}(D(\lambda_0)) = \langle k \rangle$, *where* $k \neq 0$, $D'(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$, *and* $C(\lambda_0)B(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$. *Then there is a* $\delta > 0$ *such that for all* $\lambda \in N_\delta(\lambda_0) \backslash \{\lambda_0\}$, $\mathcal{S}_\lambda \in \mathcal{L}(\mathbb{R}^n)$ *exists. There is a* C^r *locus of algebraically simple, real eigenvalues of* $\mathcal{S}_\lambda$, *denoted* $\alpha(\lambda)$, *which has a simple pole at* λ_0

$$\alpha(\lambda) = \frac{\mu}{\lambda - \lambda_0} + \phi(\lambda - \lambda_0),$$

where $\mu \neq 0$ *and the function* ϕ *is* C^{r-2} *in* $N_\delta(\lambda_0)$.

This has one fewer conditions than Theorem 1. Namely, the invertibility of $L(\lambda_0) = d_z(f \times g)(0, 0, \lambda_0)$ is needed to prove Theorem 1, but not its generalization Theorem 4. However, Theorem 4 does not tell us how the remaining $n - 1$ elements of $\sigma(\mathcal{S}_\lambda)$ behave at λ_0, nor does it provide an expression for the residue μ. However, retaining this invertibility condition permits another proof of Theorem 1, based on Lemma 4, which yields a simple expression for μ.

THEOREM 5. *Suppose that* $r \geq 3$ *and* $L \in C^r(\Lambda, \mathcal{L}(\mathbb{R}^{n+m}))$. *Suppose further that* $\mathbf{N}(D(\lambda_0)) = \langle k \rangle$, *where* $k \neq 0$, $\mathbf{N}(D(\lambda_0)^T) = \langle u \rangle$, $D'(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$, *and* $C(\lambda_0)B(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$. *In addition, suppose that* $L(\lambda_0) \in GL(\mathbb{R}^{n+m})$.

Then there is a $\delta > 0$ *such that for all* $\lambda \in N_\delta(\lambda_0) \backslash \{\lambda_0\}$, $\mathcal{S}_\lambda \in \mathcal{L}(\mathbb{R}^n)$ *exists and is invertible. There is a locus of eigenvalues,* $\alpha(\lambda) \in \sigma(\mathcal{S}_\lambda)$, *and a* C^{r-2} *function* ϕ *such that* $\alpha(\lambda) = \mu/(\lambda - \lambda_0) + \phi(\lambda - \lambda_0)$, *where*

$$\mu = -\frac{u^T C(\lambda_0) B(\lambda_0) k}{u^T D'(\lambda_0) k}.$$

For all $|\lambda - \lambda_0| < \delta$ the remaining $n-1$ eigenvalue locii of $\mathcal{S}_\lambda$ lie in some compact annulus $\mathcal{A} \subset \mathbb{C}$ which does not contain the origin. Moreover, these eigenvalues can be uniquely extended to a continuous function on $N_\delta(\lambda_0)$.

In addition, in the notation of Lemma 4

$$A_1(\lambda_0) = \lim_{\lambda \to \lambda_0} \mathcal{S}_\lambda^{-1},$$

where $\mathcal{N}(A_1(\lambda_0)) = \langle B(\lambda_0)k \rangle$ *and* $\mathcal{R}(A_1(\lambda_0)) = \left\langle C^T(\lambda_0)u \right\rangle$.

Proof. Define the matrices $S(\lambda)$ and $T(\lambda) \in \mathcal{L}(\mathbb{R}^{n+m})$,

$$T(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} I & 0 \\ C(\lambda) & D(\lambda) \end{bmatrix} \text{ and } S(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} I & B(\lambda) \\ 0 & D(\lambda) \end{bmatrix},$$

which form the Schur decomposition

$$L(\lambda) = S(\lambda) \begin{bmatrix} \mathcal{S}_\lambda & 0 \\ 0 & D(\lambda)^{-1} \end{bmatrix} T(\lambda). \tag{2.1}$$

By Lemmas 1 and 3, $\frac{d}{d\lambda} \text{det} D(\lambda)\big|_{\lambda=\lambda_0} \neq 0$ and it follows that there is some $\delta_1 > 0$ such that $\text{det} D(\lambda)$ is nonzero on $N_{\delta_1}(\lambda_0) \backslash \{\lambda_0\}$. Therefore $\mathcal{S}_\lambda$ exists on this neighborhood and is a bijection using (2.1).

As $\text{det} L(\lambda_0) \neq 0$ there is a $\delta_2 > 0$ such that $L(\lambda)^{-1} L(\lambda) \equiv L(\lambda) L(\lambda)^{-1} = I$ for all $\lambda \in N_{\delta_2}(\lambda_0)$. Let $\delta = \min(\delta_1, \delta_2)$.

Let us define the C^r map

$$\lambda \mapsto L(\lambda)^{-1} = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{bmatrix}.$$

One finds

$$A_1(\lambda)B(\lambda) + B_1(\lambda)D(\lambda) \equiv 0, \tag{2.2a}$$

$$C(\lambda)A_1(\lambda) + D(\lambda)C_1(\lambda) \equiv 0, \tag{2.2b}$$

$$C(\lambda)B_1(\lambda) + D(\lambda)D_1(\lambda) \equiv I_m, \tag{2.2c}$$

$$A(\lambda)A_1(\lambda) + B(\lambda)C_1(\lambda) \equiv I_n. \tag{2.2d}$$

By taking inverses, it follows from (2.1) that

$$\mathcal{S}_\lambda^{-1} \equiv A_1(\lambda)$$

when $\text{det} D(\lambda) \neq 0$. We shall now prove that $A_1(\lambda_0)$ has an algebraically simple eigenvalue. This will provide a curve, $\beta(\lambda)$, of eigenvalues of $A_1(\lambda)$ such that $\alpha(\lambda) = 1/\beta(\lambda) \in \sigma(\mathcal{S}_\lambda)$ and the fact that $\beta(\lambda_0) = 0$ will complete the proof.

Equation (2.2a) implies

$$A_1(\lambda_0)B(\lambda_0)k = 0.$$

Suppose that there is some $v \neq 0$ such that $A_1(\lambda_0)v = 0$. Using (2.2d) evaluated at $\lambda = \lambda_0$, we find $B(\lambda_0)C_1(\lambda_0)v = v$. Using (2.2b) we see that $D(\lambda_0)C_1(\lambda_0)v = 0$, so $C_1(\lambda_0)v = \zeta k$ for some $\zeta \in \mathbb{R}$ by the simple null-space condition. Therefore $\zeta B(\lambda_0)k = v$ and it follows that

$$\mathbf{N}(A_1(\lambda_0)) = \langle B(\lambda_0)k \rangle .$$

Now suppose $B(\lambda_0)k \in \mathbf{R}(A_1(\lambda_0))$ so that $A_1(\lambda_0)w = B(\lambda_0)k$ for some nonzero w. This yields, from (2.2b), $C(\lambda_0)A_1(\lambda_0)w + D(\lambda_0)C_1(\lambda_0)w = 0$. It follows that $C(\lambda_0)B(\lambda_0)k = -D(\lambda_0)C_1(\lambda_0)w$ and therefore $u^TC(\lambda_0)B(\lambda_0)k = 0$, a contradiction. Hence 0 is an algebraically simple eigenvalue of $A_1(\lambda_0)$ with eigenvector $B(\lambda_0)k$, therefore 0 is an algebraically simple eigenvalue of $A_1(\lambda_0)^T$.

Since, from (2.2b), $0 = (u^TC(\lambda_0)A_1(\lambda_0))^T = A_1(\lambda_0)^TC(\lambda_0)^Tu$ we find

$$\mathcal{R}(A_1(\lambda_0)) = \mathbf{R}(A_1(\lambda_0)) = \mathbf{N}(A_1(\lambda_0)^T)^\perp = \left\langle C(\lambda_0)^Tu \right\rangle^\perp.$$

From spectral perturbation results in [11] one can find a C^r, algebraically simple, real eigenvalue locus $\beta(\lambda) \in \sigma(A_1(\lambda))$ such that $\beta(\lambda_0) = 0$. Differentiating the identities

$$\beta(\lambda)v(\lambda) \equiv A_1(\lambda)v(\lambda)$$

and also (2.2a) and then setting $\lambda = \lambda_0$ yields

$$\beta'(\lambda_0) = -\frac{u^TC(\lambda_0)B_1(\lambda_0)D'(\lambda_0)k}{u^TC(\lambda_0)B(\lambda_0)k}.$$

Thus $\beta'(\lambda_0) = 0$ if and only if

$$C(\lambda_0)B_1(\lambda_0)D'(\lambda_0)k \in \mathbf{R}(D(\lambda_0)).$$

Postmultiplying (2.2c) by $D'(\lambda_0)k$, one finds

$$C(\lambda_0)B_1(\lambda_0)D'(\lambda_0)k + D(\lambda_0)D_1(\lambda_0)D'(\lambda_0)k = D'(\lambda_0)k.$$

Therefore

$$u^TC(\lambda_0)B_1(\lambda_0)D'(\lambda_0)k = u^TD'(\lambda_0)k \neq 0$$

and consequently $\beta'(\lambda_0) \neq 0$. Define $\alpha(\lambda) = 1/\beta(\lambda)$.

Since $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)\alpha(\lambda) = \mu$ and $\alpha(\lambda) = 1/\beta(\lambda) \in \sigma(A_1(\lambda)^{-1}) \equiv \sigma(S_\lambda)$, it is clear that $\mu = 1/\beta'(\lambda_0)$. Since $\mathcal{S}_\lambda = A_1(\lambda)^{-1}$ for $\lambda \neq \lambda_0$, and $A_1(\lambda_0)$ has an algebraically simple zero eigenvalue, it has $n-1$ nonzero eigenvalues whose reciprocals provide the elements of $\mathbb{C}$ required to remove the $n-1$ singularities in $\sigma(\mathcal{S}_\lambda)$. ∎

COROLLARY 1. *Under the conditions of Theorem* 5, *if* $\lambda \in N_\delta(\lambda_0)\backslash\{\lambda_0\}$, *then* $\#\sigma(M, L(\lambda)) = n$, *however,* $\#\sigma(M, L(\lambda_0)) = n - 1$.

Proof. Use Lemma 4 to count the number of nonzero eigenvalues of $A_1(\lambda)$ according to algebraic multiplicity. ∎

THEOREM 6. *If* (M, L) *is a regular pencil, then* $\mathrm{ind}(M, L) = 1$ *if and only if* $\det D \neq 0$.

Proof. If $\det D \neq 0$, then it is simple to show that[1] $\mathrm{ind}(M, L) = 1$.

Now suppose that (M, L) is regular and $\mathrm{ind}(M, L) = 1$. Hence there are matrices P_{ij} and Q_{ij}, for $i = 1, 2$ and $j = 1, 2$, which form nonsingular maps P and Q, such that for some n_1 and n_2 with $n_1 + n_2 = n + m$,

$$PMQ = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0_{n_2} \end{bmatrix} \stackrel{\text{def}}{=} \mathbf{M}'. \tag{2.3}$$

Now $m = \dim \mathbf{N}(M) = \dim \mathbf{N}(PMQ) = \dim \mathbf{N}(\mathbf{M}') = n_2$ and it follows that $n = n_1$.

[1] See the proof of Theorem 7 for details.

Using (2.3) we find that $P_{11}Q_{11} = I_n$, $Q_{12} = 0$, and $P_{21} = 0$. From this it is clear that $\det P = \det P_{11} \det P_{22}$ and $\det Q = \det Q_{22}/\det P_{11}$. Also, for some $\hat{C} \in \mathcal{L}(\mathbb{R}^n)$

$$\begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P_{11}^{-1} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} \hat{C} & 0 \\ 0 & I_m \end{bmatrix}. \tag{2.4}$$

It follows that $P_{22}DQ_{22} = I_m$ and therefore $\det D \neq 0$. □

COROLLARY 2. *If (M, L) is regular and $\operatorname{ind}(M, L) = 1$ then, there is a $\delta > 0$ such that if $\|L - L'\| < \delta$, then (M, L') is regular and $\operatorname{ind}(M, L') = 1$.*

In what follows we shall use $P(M, L)Q$ to denote the pencil (PMQ, PLQ). Consider the following example which shows various KNFs of a class of matrix pencil which has the same form as (M, L).

Example 1. Consider the matrix pencil

$$(M, L) \stackrel{\text{def}}{=} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

If $d \neq 0$, this pencil has index-1. If $d = 0$, but $bc \neq 0$, it has index-2. If $|d| + |bc| = 0$, it is not regular.

Clearly, if $d \neq 0$, the normal form is given by

$$\begin{bmatrix} 1 & -b/d \\ 0 & 1/d \end{bmatrix} (M, L) \begin{bmatrix} 1 & 0 \\ -c/d & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a - bc/d & 0 \\ 0 & 1 \end{bmatrix} \right).$$

If $d = 0$ but $bc \neq 0$, then multiplying by the inverse of the second matrix within this pencil gives the normal form

$$L^{-1}(M, L)I = \left(\begin{bmatrix} 0 & 0 \\ 1/b & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

Within the definition of KNF this corresponds to the vector subspace U being trivial, namely, $U = \emptyset$.

The following theorem generalizes Example 1 to matrix pencils over $\mathbb{R}^{n+m}$.

THEOREM 7. *Suppose that $n \geq 2, \det L(\lambda_0) \neq 0, \mathbf{N}(D(\lambda_0)) = \langle k \rangle, C(\lambda_0)B(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$, and $D'(\lambda_0)k \notin \mathbf{R}(D(\lambda_0))$. There is some $r_0 > 0$ such that*

$$\operatorname{ind}(M, L(\lambda)) = \begin{cases} 1 & \text{if} \quad \lambda \in N_{r_0}(\lambda_0) \backslash \{\lambda_0\}, \\ 2 & \text{if} \quad \lambda = \lambda_0. \end{cases}$$

Moreover, the KNF of $(M, L(\lambda))$ is given as follows. If $\lambda \in N_{r_0}(\lambda_0)\backslash\{\lambda_0\}$, then there are mappings P' and Q' such that

$$P'MQ' = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \text{ and } P'L(\lambda)Q' = \begin{bmatrix} \mathcal{S}_\lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

There are mappings P, Q, and L_0 such that

$$PMQ = \begin{bmatrix} I_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_0 & 0 \end{bmatrix} \text{ and } PL(\lambda_0)Q = \begin{bmatrix} L_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

where $C_0 : \mathbb{R} \to \mathbb{R}^m$ is a linear map such that $C_0(1) = k$. If we denote $\mathbf{N}(D(\lambda_0)^T) = \langle u \rangle$ and write $U = \left\langle C(\lambda_0)^T u \right\rangle^\perp$, then $L_0 \in GL(U)$. The mappings PMQ and $PL(\lambda_0)Q$ are elements of $\mathcal{L}(U \oplus \mathbb{R} \oplus \mathbb{R}^m)$.

Proof. From Theorem 5 we know that $\mathcal{N}(A_1(\lambda_0)) = \langle B(\lambda_0)k\rangle$ and that there is an $r_0 > 0$ such that if $\lambda \in N_{r_0}(\lambda_0)$, $\det D(\lambda) = 0$ if and only if $\lambda = \lambda_0$. If $\lambda \neq \lambda_0$ is in this neighborhood, define

$$P' = \begin{bmatrix} I_n & -B(\lambda)D(\lambda)^{-1} \\ 0 & I_m \end{bmatrix} \text{ and } Q' = \begin{bmatrix} I_n & 0 \\ -D(\lambda)^{-1}C(\lambda) & D(\lambda)^{-1} \end{bmatrix}.$$

A multiplication shows that the KNF is given by

$$P'MQ' = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \text{ and } P'L(\lambda)Q' = \begin{bmatrix} (A - BD^{-1}C)(\lambda) & 0 \\ 0 & I_m \end{bmatrix}.$$

Hence $\operatorname{ind}(M, L(\lambda)) = 1$ for all $\lambda \in N_{r_0}(\lambda_0)\backslash\{\lambda_0\}$.

We now calculate $\operatorname{ind}(M, L(\lambda_0))$. We adopt the same notation as Theorem 5 and drop the reference to λ_0 for brevity.

Write $\mathcal{R} \stackrel{\text{def}}{=} \mathbf{R}(A_1) = \langle C^T u\rangle^{\perp}$, so that $\mathbb{R}^n = \mathcal{R} \oplus \langle Bk\rangle$ and $\dim \mathcal{R} = n-1$. Define the restriction

$$A_0 \stackrel{\text{def}}{=} A_1|_{\mathcal{R}} \in GL(\mathcal{R}).$$

Since 0 is an algebraically simple eigenvalue of A_1, there is an invertible map $J : \mathcal{R} \oplus \mathbb{R} \to \mathbb{R}^n$ such that

$$J^{-1}A_1J = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{R} \oplus \mathbb{R}).$$

Therefore

$$J \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \gamma Bk$$

for some $\gamma \in \mathbb{R}$. Without the loss of any generality, we may normalize k to ensure that $\gamma = 1$.

If we premultiply (M, L) by L^{-1} we obtain the pencil

$$L^{-1}(M, L) = \left(\begin{bmatrix} A_1 & 0 \\ C_1 & 0 \end{bmatrix}, I_{n+m} \right) \in \mathcal{L}(\mathbb{R}^{n+m}) \times \mathcal{L}(\mathbb{R}^{n+m}).$$

Conjugating this pencil with the map $\mathcal{J} : \mathcal{R} \oplus \mathbb{R} \oplus \mathbb{R}^m \to \mathbb{R}^{n+m}$ given by

$$\mathcal{J} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} Jr \\ s \end{bmatrix} \quad (r \in \mathcal{R} \oplus \mathbb{R}, s \in \mathbb{R}^m)$$

yields

$$\mathcal{J}^{-1}(L^{-1}M, I_{n+m})\mathcal{J} = \left(\begin{bmatrix} A_0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1^0 & C_1^1 & 0 \end{bmatrix}, I \right) \in \mathcal{L}(\mathcal{R} \oplus \mathbb{R} \oplus \mathbb{R}^m) \times \mathcal{L}(\mathcal{R} \oplus \mathbb{R} \oplus \mathbb{R}^m),$$

where $C_1^0 = C_1J|_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^m$ and $C_1^1 = C_1J|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}^m$. Hence, for $\eta \in \mathbb{R}$, $C_1^1(\eta) = \eta C_1^1(1)$ and

$$C_1^1(1) = C_1J \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1Bk = k.$$

Here I is used to denote the identity on $\mathcal{R} \oplus \mathbb{R} \oplus \mathbb{R}^m$.

Conjugate $(\mathcal{J}^{-1}L^{-1}M\mathcal{J}, I)$ by

$$\begin{bmatrix} I_{\mathcal{R}} & 0 & 0 \\ 0 & 1 & 0 \\ -C_1^0 A_0^{-1} & 0 & I_m \end{bmatrix} \in GL(\mathcal{R} \oplus \mathbb{R} \oplus \mathbb{R}^m).$$

Now pre- or postmultiply the resulting pencil by

$$\begin{bmatrix} A_0^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix} \in GL(\mathcal{R} \oplus \mathbb{R} \oplus \mathbb{R}^m).$$

This provides nonsingular transformations P and Q which, when applied simultaneously to M and L, yield

$$PMQ = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_1^1 & 0 \end{bmatrix} \text{ and } PL(\lambda_0)Q = \begin{bmatrix} A_0^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

Comparing (PMQ, PLQ) to the KNF, we see that the nilpotent mapping N is

$$\begin{bmatrix} 0 & 0 \\ C_1^1 & 0 \end{bmatrix} \in L(\mathbb{R} \oplus \mathbb{R}^m).$$

We see that this is nonzero because $C_1^1(1) = k$. However, $N^2 = 0$ so that $\operatorname{ind}(M, L(\lambda_0)) = 2$. □

The following theorem shows that the index of (M, L) is not always stable to perturbations in L. Of course, we can easily see that the index of (M, L) is not stable to perturbations in M because $\operatorname{ind}(M(\epsilon), L(\lambda)) = 0$ for $\lambda \neq \lambda_0$ and $\epsilon \neq 0$, yet $\operatorname{ind}(M, L(\lambda)) = 1$.

Let us note that one can easily formulate an analogy of Theorem 7 to cover the case when $n = 1$. We omit this for brevity.

Example 2. Let $\phi_i(t)$ be smooth functions and consider the DAE

$$\begin{aligned} \dot{x}(t) &= y_1(t) + \phi_1(t), \\ 0 &= x(t) + y_2(t) + \phi_2(t), \\ 0 &= \lambda y_1(t) + y_2(t) + \phi_3(t). \end{aligned}$$

This can be written in the form $M\dot{z}(t) = L(\lambda)z(t) + f(t)$ such that $(M, L(\lambda))$ satisfies the SIB theorem at $\lambda = 0$. If $\lambda \neq 0$, the solution is easily seen not to depend on any of the derivatives of ϕ_i. We find that the solution of this DAE when $\lambda = 0$ is given by

$$x = \phi_3 - \phi_2, \ y_1 = \dot{\phi}_3 - \dot{\phi}_2 - \phi_1, \ y_2 = -\phi_3.$$

The dependence of the solution on the first derivatives of ϕ_i is to be expected because the index of $(M, L(0))$ is 2.

3. Unfolding the SIB. We continue with the following definition.

DEFINITION 1. *A* singular Hopf curve *is the graph of a continuous function* $\mathcal{H} = \{(\epsilon, \lambda_0(\epsilon)) \in \mathbb{R}^2 : \epsilon \in (0, \epsilon_0)\}$ *such that* $(M(\epsilon), L(\lambda))$ *has a purely imaginary eigenvalue* $i\omega(\epsilon)$ *for all* $(\epsilon, \lambda) \in \mathcal{H}$ *and* $\lim_{\epsilon \to 0_+} \omega(\epsilon) = \infty$. *A point on* $\mathcal{H}$ *is called a Hopf point.*

Let us remark that the existence of a Hopf curve is not sufficient to determine the existence of periodic solutions in an ODE whose linearization is $(M(\epsilon), L(\lambda))$ without further transversality assumptions such as those found in the Hopf bifurcation theorem [12, 1]. Nevertheless, we proceed with the following theorem which provides the *necessary* ingredient for Hopf bifurcation.

THEOREM 8. *Suppose that the matrix pencil $(M, L(\lambda))$ satisfies the hypotheses of Theorem 5 and, adopting the notation of that theorem, $u^T k \neq 0$ and*

$$\frac{u^T C(\lambda_0) B(\lambda_0) k}{u^T k} = -\omega_0^2 < 0. \tag{3.1}$$

Then there is an $\epsilon_0 > 0$ and unique C^r functions $\lambda_0 : (0, \epsilon_0) \to \mathbb{R}$ and $\omega_0 : (0, \epsilon_0) \to (0, \infty)$ such that $(M(\epsilon), L(\lambda_0(\epsilon)))$ has a pair of purely imaginary, algebraically simple eigenvalues $\pm i\omega_0(\epsilon)$. Moreover, $\lambda_0(\epsilon) = \lambda_0 + O(\epsilon)$ and $\epsilon^{1/2}\omega_0(\epsilon) = \omega_0 + O(\epsilon)$ as $\epsilon \to 0_+$.

Proof. If $z = (x, y) \in \mathbb{R}^{n+m}$, let $\Pi_1 z = x$ and $\Pi_2 z = y$ be the projections of z onto coordinate components. Let $N(\epsilon) = \epsilon M(\epsilon)^{-1}$. Write $\rho = \omega^2 \epsilon$ and seek a solution of the augmented system

$$[\rho M(\epsilon) + L(\lambda) N(\epsilon) L(\lambda)]\, z = 0, \tag{3.2a}$$

$$u^T C(\lambda)(\Pi_1 z) - 1 = 0, \tag{3.2b}$$

$$k^T (\Pi_2 z) - 1 = 0. \tag{3.2c}$$

It is clear that a solution of (3.2) provides a purely imaginary pair of eigenvalues of $(M(\epsilon), L(\lambda))$. Write (3.2) as $F(z, \lambda, \rho, \epsilon) = 0$, where $F : \mathbb{R}^{n+m} \times \mathbb{R}^3 \to \mathbb{R}^{n+m} \times \mathbb{R}^2$ is a C^r mapping. Set

$$\rho_0 = \omega_0^2, x_0 = \frac{B(\lambda_0) k}{u^T C(\lambda_0) B(\lambda_0) k}$$

and let y_0 be the unique solution of

$$\frac{-k}{u^T k} + \frac{C(\lambda_0) B(\lambda_0) k}{u^T C(\lambda_0) B(\lambda_0) k} + D y_0 = 0,\ k^T y_0 = 1.$$

It follows that $F((x_0, y_0), \lambda_0, \rho_0, 0) = 0$.

Now $d_{z,\lambda,\rho} F(z, \lambda, \rho, \epsilon) \in \mathcal{L}(\mathbb{R}^{n+m} \times \mathbb{R}^2)$ is given by the matrix

$$\begin{bmatrix} \rho M(\epsilon) + L(\lambda) N(\epsilon) L(\lambda) & L'(\lambda) N(\epsilon) L(\lambda) + L(\lambda) N(\epsilon) L(\lambda)' & M(\epsilon) z \\ u^T C(\lambda) \Pi_1 \cdot & 0 & 0 \\ k^T \Pi_2 \cdot & 0 & 0 \end{bmatrix}. \tag{3.3}$$

Suppose that $d_{z,\lambda,\rho} F(z_0, \lambda_0, \rho_0, 0)[h, \alpha, \beta] = (0, 0, 0)$ for some $(h, \alpha, \beta) \in \mathbb{R}^{n+m} \times \mathbb{R}^2$, and write $h_1 = \Pi_1 h$ and $h_2 = \Pi_2 h$. Then, using the fact that $L(\lambda_0)$ is invertible and removing the reference to λ_0 for brevity,

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \rho_0 A_1 & 0 \\ \rho_0 C_1 + C & D \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &\quad + \alpha \left\{ L^{-1} L' \begin{bmatrix} 0 \\ \frac{-\rho_0}{u^T C B k} k \end{bmatrix} + \begin{bmatrix} 0 \\ C' x_0 + D' y_0 \end{bmatrix} \right\} + \beta \begin{bmatrix} 0 \\ \frac{1}{u^T C B k} k \end{bmatrix}, \end{aligned} \tag{3.4}$$

where $u^T C h_1 = 0$ and $k^T h_2 = 0$.

Therefore,

$$\rho_0 A_1 h_1 + \alpha(A_1 B' + B_1 D')k = 0$$

and, premultiplying this by $u^T C$ and using the fact that $A_1^T C^T u = 0$, we find $0 = \alpha u^T C B_1 D' k$. Since $CB_1 = I - DD_1$, it follows that

$$0 = \alpha u^T C B_1 D' k = \alpha u^T (I - DD_1) D' k = \alpha u^T D' k$$

and $\alpha = 0$ follows. Hence there is a $\theta_1 \in \mathbb{R}$ such that $h_1 = \theta_1 Bk$, so that $0 = u^T C h_1 = \theta_1 u^T CBk$ and $h_1 = 0$ also follows.

Finally, therefore, $Dh_2 + \frac{\beta}{u^T CBk} k = 0$. As $u^T k \neq 0$, $\beta = 0$ must follow. Therefore, there is a $\theta_2 \in \mathbb{R}$ such that $h_2 = \theta_2 k$. However, $k^T h_2 = 0$ implies that $h_2 = 0$. Hence, the derivative $d_{z,\lambda,\rho} F((x_0, y_0), \lambda_0, \rho_0, 0)$ is a bijection and the result follows by the implicit function theorem. □

The sign condition (3.1) within Theorem 8 can be replaced by

$$\frac{u^T C(\lambda_0) B(\lambda_0) k}{u^T k} = \omega_0^2 > 0$$

and the conclusions of the above theorem remain true, but instead for the matrix pencil $(M(-\epsilon), L(\lambda))$.

The following system of ODEs from [13] probes Theorem 8 quite effectively. It is a model of cooperative phenomena which occur in oxygen-hemoglobin reactions taking place in the bloodstream (details of this can be found in [13, p. 119]). The model is

$$\frac{du}{dt} = -u + (u - a_3 u + a_1)v_1 + (a_4 + u)v_2, \tag{3.5a}$$

$$\epsilon \frac{dv_1}{dt} = u - (u + a_3 u + a_1 + a_2)v_1 + (a_4 + a_5 - u)v_2, \tag{3.5b}$$

$$\epsilon \frac{dv_2}{dt} = a_3 u v_1 - (a_4 + a_5)v_2, \tag{3.5c}$$

where $u(0) = 1$ and $v_1(0) = v_2(0) = 0$.

The author of [13] only considers the behavior of (3.5) for large time by finding the projection of the flow onto the one dimensional slow manifold. Instead, suppose we consider the stability of the trivial equilibrium which exists for all parameter values a_i. The linearization of (3.5) at this equilibrium point is the matrix

$$M(\epsilon)^{-1} L(\lambda, \mu) \stackrel{\text{def}}{=} \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \epsilon^{-1} & 0 \\ 0 & 0 & \epsilon^{-1} \end{array}\right] \left[\begin{array}{c|cc} -1 & a_1 & a_4 \\ \hline 1 & -\lambda & \mu \\ 0 & 0 & -\mu \end{array}\right], \tag{3.6}$$

where $\lambda = a_1 + a_2$ and $\mu = a_4 + a_5$. We consider the stability properties of the origin when λ and μ vary, with the other parameters fixed and ϵ small.

For fixed $\lambda \neq 0$ and varying μ, a slow-manifold singularity is encountered by the trivial equilibrium at $\mu = 0$. However, $\omega_0 = 0$ in the notation of Theorem 8. One finds on inspection that a Hopf curve does not emanate from the point $(\mu, \epsilon) = (0, 0)$.

However, for a fixed $\mu \neq 0$ and varying λ, Theorem 8 is satisfied at $\lambda = 0$, provided $a_1 < 0$. A curve of Hopf points exists, and is given by the graph of λ_0. Theorem 8 indicates that

$$\omega_0(\epsilon) = \epsilon^{-1/2}\sqrt{-a_1} + O(\epsilon^{1/2}), \ \lambda_0(\epsilon) = -\epsilon + O(\epsilon^2).$$

A simple calculation shows that

$$\omega_0(\epsilon) = \epsilon^{-1/2}\sqrt{a_1 + \epsilon} \text{ and } \lambda_0(\epsilon) = -\epsilon.$$

The value of μ has no effect on the appearance of a Hopf curve in (λ, ϵ)-space, but μ must be nonzero for Theorem 8 to apply. We therefore see that Theorem 8 provides sufficient, but not necessary conditions for the existence of a Hopf curve. Namely, the invertibility of $L(\lambda_0)$, which is used to prove Theorems 5 and 8, is not needed for a Hopf curve to exist.

REFERENCES

[1] V. I. Arnold, V. S. Afrajmovich, Y. S. Il'yashenko, and L. P. Shil'nikov, *Bifurcation Theory and Catastrophe Theory*, Springer-Verlag, New York, 1999.

[2] S. M. Baer and T. Erneux, *Singular Hopf bifurcation to relaxation oscillations*, SIAM J. Appl. Math. II, 52 (1992), pp. 1651–1664.

[3] R. E. Beardmore, *Stability and bifurcation properties of index-1 DAEs*, Numer. Algorithms, 19 (1998), pp. 43–53.

[4] R. E. Beardmore, *Flows near singular equilibria in differential-algebraic equations*, SIAM J. Matrix Anal. Appl., submitted.

[5] E. Benoit, *Systèmes lents-rapides dans R^3 et leurs canards*, Astérisque, (1983), pp. 159–191.

[6] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, North-Holland, Amsterdam, 1989.

[7] S. Campbell, R. Hollenbeck, and W. Marszalek, *Mixed symbolic-numeric computations with general DAE I: System properties*, Numer. Algorithms, 19 (1998), pp. 73–84.

[8] F. Dumortier and R. Roussarie, *Canard cycles and center manifolds*, Memoirs of the AMS, 121 (1996), p. 1.

[9] F. R. Gantmacher, *Theory of Matrices*, Chelsea Publishing Co., New York, 1977.

[10] G. Golub and C. F. V. Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 1989.

[11] T. Kato, *Perturbation Theory for Linear Operators*. Reprint of the 1980 Edition, Springer-Verlag, New York, 1995.

[12] J. E. Marsden and M. McCracken, *The Hopf Bifurcation Theorem and its Applications*, Springer-Verlag, New York, 1976.

[13] J. D. Murray, *Mathematical Biology*, 2nd ed., Biomathematics 19, Springer-Verlag, Berlin, 1980.

[14] R. Riaza, S. L. Campbell, and W. Marszalek, *On singular equilibria of index-1 DAEs*, Circuits Systems Signal Process, (2000), pp. 131–157.

[15] M. Stiefenhofer, *Singular perturbation with limit points in the fast dynamics*, Z. Zngew. Math. Phys., 49 (1998), pp. 730–758.

[16] V. Venkatasubramanian, H. Schattler, and J. Zaborszky, *A Stability Theory of Large Differential Algebraic Systems, a Taxonomy, No. SSM* 9201 *Part* 1, Technical Report, Washington University, St. Louis, MO, 1992.

[17] V. Venkatasubramanian, H. Schattler, and J. Zaborszky, *Dynamics of large constrained nonlinear systems—a taxonomy theory*, Proceedings of the IEEE, 83 (1995), pp. 1530–1558.

[18] A. K. Zvonkin and M. A. Shubin, *Non-standard analysis and singular perturbations of ordinary differential equations*, Russian Math. Surveys, 39 (1984), pp. 69–131.